ECE 206 – Spring 2018
Practice exam (Exam from Fall 2017)

1. Show that $f(z) = \mathrm{Log}(z)$ is analytic on the domain $D = \{re^{j\theta} : r > 0,\ -\pi < \theta < \pi\}$ in $\mathbb{C}$, and further show that $f'(z) = \frac{1}{z}$ on this domain. (Hint: use polar representation.)

2. Let Γ be the path first along the straight line from $z = -j$ to $z = 1 - j$, followed by the straight line from $z = 1 - j$ to $z = 1 + \sqrt{3}j$. Evaluate $\displaystyle\int_\Gamma \frac{1}{z}\,dz$.

3. There are two possible Laurent series expansions about $z = 0$ for the function $f(z) = \frac{1}{z^2(1-z)}$. Write down both and state the region where each is valid.

4. Let $a > 0$ be a constant and let D be the semi-circular region in $\mathbb{R}^2$ defined by
$$D = \{(x, y) : x^2 + y^2 \le a^2,\ y \ge 0\},$$
with boundary curve ∂D oriented clockwise, and define the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\boldsymbol{F}(x, y) = (xe^x, xy^2)$. Sketch the region D. Using an applicable theorem, evaluate $\displaystyle\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r}$.

5. (a) Determine the circulation of $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{F}(x, y, z) = (2yz, 0, xy)$ around the curve Γ parameterized by $\boldsymbol{\gamma}(t) = (2\sin t,\ 2\cos t,\ 1)$ for $t \in [0, 2\pi]$.
   (b) Determine the flux of the vector field $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{G}(x, y, z) = (x, y, -2z)$ through the surface Σ, which is the portion of the paraboloid $z = \frac{1}{2}(x^2 + y^2) - 1$ that lies below the plane $z = 1$ (where $\hat{\boldsymbol{n}}$ is the outward unit normal).
   (c) If all went well, your answers in (a) and (b) should match. What is the relationship between $\boldsymbol{F}$ and $\boldsymbol{G}$? Explain using a theorem from vector calculus. A sketch may help your explanation.

6. (a) Faraday's law states that: "*the circulation of the electric flux around the perimeter of a surface is equal to the negative time rate of change of the magnetic flux through the surface.*" Write down the integral form of Faraday's law. Then use theorems of vector calculus to derive the partial differential equation form of the law.
   (b) Refer to Maxwell's equations to explain why a magnetic potential $\boldsymbol{A}$ exists. Then show that the time varying vector field $\boldsymbol{E} + \dfrac{\partial \boldsymbol{A}}{\partial t}$ is conservative.
   (c) Given a surface Σ with boundary curve $\partial\Sigma$, show that the magnetic flux through Σ can be computed using a line integral around $\partial\Sigma$.

7. For each $f(z)$ below, evaluate $\displaystyle\oint_\Gamma f(z)\,dz$, where Γ is the positively oriented circle $|z - j| = 2$. Be sure to explain your approach (i.e. state which theorem you use) in each case.
   (a) $f(z) = \dfrac{z^2 \cosh z}{(z - 10)^3}$
   (b) $f(z) = z^4 \sin\left(\frac{1}{jz}\right)$

(c) $f(z) = \dfrac{e^{jz}}{(z^2+4)^2}$

8. (a) Suppose $f(z)$ is analytic except at a finite number of singularities in the upper half plane. Explain mathematically how the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ can be evaluated using the residue theorem. (Hint: Start with an appropriate contour Γ. You should mention a condition on $|f(z)|$.)

   (b) Using the method described above, evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$.

9. Suppose $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field of the form $\boldsymbol{F}(x,y,z) = f(r)\boldsymbol{r}$, where $\boldsymbol{r} = (x,y,z)$, $r = \|\boldsymbol{r}\| = \sqrt{x^2+y^2+z^2}$, and f is an arbitrary function.

   (a) Use the product rule for curl to show that $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

   (b) If Γ_1 is a closed curve in $\mathbb{R}^3$, what is $\displaystyle\oint_{\Gamma_1} \boldsymbol{F}\cdot d\boldsymbol{r}$?

   (c) By part (a), there exists a scalar potential Ψ such that $\boldsymbol{F} = \nabla\Psi$. Based on the form of $\boldsymbol{F}$, we can assume that Ψ depends only on r (i.e., $\Psi = g(r)$ for some function g). Find a relation between f and g.

   (d) Use part (c) to find Ψ in the case when $\boldsymbol{F} = -\dfrac{\boldsymbol{r}}{r^3}$. Let Γ_2 be a curve that starts at $(1,2,3)$ and ends at $(-1,-2,-3)$. Evaluate $\displaystyle\int_{\Gamma_2} \boldsymbol{F}\cdot d\boldsymbol{r}$. Interpret your result physically.

10. (a) Let Ω be a region in $\mathbb{R}^3$, with closed boundary surface $\partial\Omega$, and let $\boldsymbol{c} : \mathbb{R}^3 \to \mathbb{R}^3$ be a *constant* vector field. Show that the flux of $\boldsymbol{c}$ through $\partial\Omega$ is zero.

    (b) A function f is *harmonic* if it satisfies the Laplace equation $\nabla^2 f = 0$. Recall that $\nabla^2 f = \nabla\cdot(\nabla f)$. Let Ω be a region in $\mathbb{R}^3$ with closed boundary surface $\partial\Omega$. Show that if a function f is harmonic everywhere in Ω, and $f = 0$ on $\partial\Omega$, then $f = 0$ everywhere in Ω.

    Hint: Start by showing that $\displaystyle\iint_{\partial\Omega} (f\nabla f)\cdot\hat{\boldsymbol{n}}\,dS = \iiint_{\Omega} (f\nabla^2 f + \|\nabla f\|^2)\,dV$.