Practice Final Solutions

1) On $D = \{re^{j\theta} : r > 0, \; -\pi < \theta < +\pi\}$, Log is defined as

$$\mathrm{Log}(z) = \log(re^{j\theta}) = \ln(r) + j\theta$$
$$= u(r,\theta) + j\, v(r,\theta)$$

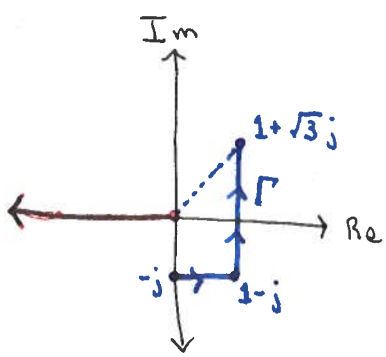
where the components are $u(r,\theta) = \ln(r)$ and $v(r,\theta) = \theta$.
We have

$$u_r = \frac{1}{r} = \frac{1}{r} v_\theta \qquad (\text{since } v_\theta = 1)$$
$$u_\theta = 0 \text{ and } v_r = 0$$

So the Cauchy-Riemann equations in polar form are satisfied, for all $z \in D$. Hence Log is analytic on D with derivative

$$f'(z) = e^{-j\theta}(u_r + j v_r) = e^{-j\theta}\left(\frac{1}{r} + 0\right)$$
$$= \frac{1}{re^{j\theta}} = \boxed{\frac{1}{z}}.$$

2)



The contour Γ has initial point $-j$ and terminal point $1+\sqrt{3}j$, and the contour stays entirely in the domain D. (The only part of $\mathbb{C}$ that is not part of D is the negative real axis indicated in red).

Since the function $f(z) = \mathrm{Log}(z)$ is analytic everywhere on D and has derivative $f'(z) = \frac{1}{z}$, we may compute the integral as

$$\int_\Gamma \frac{1}{z}\, dz = \int_\Gamma f'(z)\, dz = f(1+\sqrt{3}j) - f(-j)$$
$$= \mathrm{Log}(2e^{j\pi/3}) - \mathrm{Log}(e^{-j\pi/2})$$
$$= \ln 2 + j\frac{\pi}{3} + j\frac{\pi}{2}$$
$$= \boxed{\ln 2 + j\frac{5\pi}{6}},$$

where we use the fundamental theorem of complex integration and the fact that $1+\sqrt{3}j = 2e^{j\pi/3}$.

3) $f(z) = \dfrac{1}{z^2(1-z)}$

i) We can expand f as

$$f(z) = \frac{1}{z^2}\frac{1}{(1-z)} = \frac{1}{z^2}\left(1+z+z^2+z^3+\cdots\right)$$
$$= \frac{1}{z^2}+\frac{1}{z}+1+z^2+z^3+\cdots$$
$$= \frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty} z^n,$$

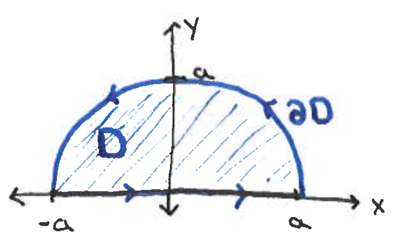
which is valid for $0<|z|<1$.

ii) We can expand f as

$$f(z) = \frac{1}{z^2}\left(\frac{1}{\frac{1}{z}-1}\frac{1}{z}\right) = -\frac{1}{z^3}\frac{1}{1-\frac{1}{z}}$$
$$= -\frac{1}{z^3}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$
$$= -\sum_{n=3}^{\infty}\frac{1}{z^n}$$

which is valid for $|\frac{1}{z}|<1$ or equivalently $|z|>1$.

4) $a>0$, $D=\{(x,y) : x^2+y^2 \leq a,\ y \geq 0\}$



$\vec{F}(x,y) = (F_1(x,y),\ F_2(x,y))$

where $F_1(x,y) = xe^x$

$F_2(x,y) = xy^2$.

Since $\vec{F}$ is C^1 vector field, we may use Green's Theorem to find that

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$
$$= \iint_D y^2\, dA$$

$$= \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} y^2 \, dy \, dx$$

$$= \int_{-a}^{a} \left(\frac{1}{3} y^3 \Big|_{y=0}^{\sqrt{a^2-x^2}} \right) dx$$

$$= \frac{1}{3} \int_{-a}^{a} (a^2-x^2)^{3/2} \, dx \qquad \begin{aligned} x &= a\sin u \\ dx &= a\cos u \, du \end{aligned}$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\pi/2} (a^2 - a^2\sin^2 u)^{3/2} \, a\cos u \, du$$

$$= \frac{1}{3} a^4 \int_{-\pi/2}^{\pi/2} \underbrace{(1-\sin^2 u)^{3/2}}_{(\cos^2 u)^{3/2}} \cos u \, du$$

$$= \frac{1}{3} a^4 \int_{-\pi/2}^{\pi/4} \cos^4 u \, du$$

$$= \ldots$$

$$= \boxed{a^4 \frac{\pi}{8}}.$$

5) a) $\vec{F}(x,y,z) = (2yz, 0, xy)$

$\vec{\gamma}(t) = (2\sin t,\ 2\cos t,\ 1), \qquad t \in [0, 2\pi]$.

$\vec{\gamma}\,'(t) = (2\cos t, -2\sin t, 0)$.

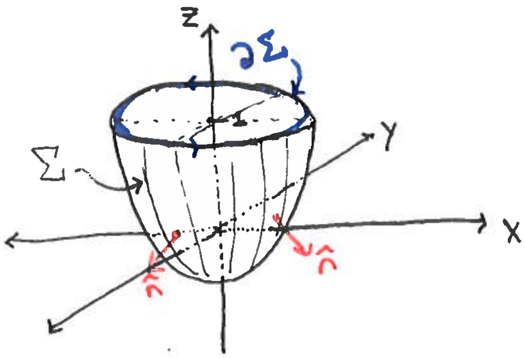
$\vec{F}(\vec{\gamma}(t)) = (4\cos t,\ 0,\ 4\cos t \sin t)$

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \int_{0}^{2\pi} \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t) \, dt$$

$$= \int_{0}^{2\pi} 8\cos^2 t \, dt = \boxed{8\pi}.$$

b) $\vec{G}(x,y,z) = (x, y, -2z)$

Σ is defined by $z = \frac{1}{2}(x^2+y^2) - 1$ below $z = 1$.



Parameterize as

$x = r\cos\theta$

$y = r\sin\theta$

$z = \frac{1}{2}(r^2) - 1$

for $\theta \in [0, 2\pi]$, $r \in [0, 2]$.

$\vec{\Phi} = \left(r\cos\theta,\ r\sin\theta,\ \frac{r^2}{2} - 1\right)$

$\frac{\partial\vec{\Phi}}{\partial r} = (\cos\theta,\ \sin\theta,\ r)$

$\frac{\partial\vec{\Phi}}{\partial\theta} = (-r\sin\theta,\ r\cos\theta,\ 0)$

$\vec{n} = \frac{\partial\vec{\Phi}}{\partial\theta} \times \frac{\partial\vec{\Phi}}{\partial r}$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & r \end{vmatrix}$

$= \hat{\imath}\, r^2\cos\theta + \hat{\jmath}\, r^2\sin\theta - \hat{k}(r\sin^2\theta + r\cos$

$= r^2(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta) - r\hat{k}$

$\uparrow$ outward pointing.

$\vec{G}(\vec{\Phi}(r,\theta)) = (r\cos\theta, r\sin\theta, 2-r^2)$

$\vec{G}\cdot\vec{n} = r^3\cos^2\theta + r^3\sin^2\theta - r(2-r^2)$

$= r^3 + r^3 - 2r = 2r(r^2-1)$

Thus

$$\iint_{\Sigma} \vec{G}\cdot\hat{n}\, dA = \iint_{\Sigma_{r,\theta}} \vec{G}(\vec{\Phi}(r,\theta)) \cdot \cancel{\frac{\vec{n}}{\|\vec{n}\|}}\ \cancel{\|\vec{n}\|}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 2r(r^2-1)\, dr$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(r^2-1)^2\right]_{r=0}^{2} d\theta$$

$$= 2\pi\, \frac{1}{2}\left[3^2 - 1^2\right] = 8\pi.$$

c) Note that $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & 0 & xy \end{vmatrix}$$

$$= \hat{\imath}\, x + \hat{\jmath}\,(2y - y) + \hat{k}\,(-2z)$$

$$= (x, y, -2z) = \vec{G}.$$

By Stokes' theorem,

$$\iint_{\Sigma} \nabla \times \vec{F} \cdot \hat{n}\, dA = \oint_{\partial \Sigma} \vec{F} \cdot d\vec{r} = \int_{\Gamma} \vec{F} \cdot d\vec{r}$$

since $\partial \Sigma = \Gamma$.

6) a) Integral form of Faraday's Law: For any surface Σ in $\mathbb{R}^3$,

$$\oint_{\partial \Sigma} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_{\Sigma} \vec{B} \cdot \hat{n}\, dA.$$

By Stokes' Theorem, $\oint_{\partial \Sigma} \vec{E} \cdot d\vec{r} = \iint_{\Sigma} (\nabla \times \vec{E}) \cdot \hat{n}\, dA$.

Thus

$$0 = \oint_{\partial \Sigma} \vec{E} \cdot d\vec{r} + \frac{\partial}{\partial t} \iint_{\Sigma} \vec{B} \cdot \hat{n}\, dA$$

$$= \iint_{\Sigma} \left((\nabla \times \vec{E}) \cdot \hat{n} + \frac{\partial}{\partial t} \vec{B} \cdot \hat{n} \right) dA$$

$$= \iint_{\Sigma} \left(\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} \right) \cdot \hat{n}\, dA.$$

Since this is true for any possible choice of surface Σ, by the du Bois-Reymond Lemma it follows that

$$\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = \vec{0} \quad \text{everywhere,}$$

or $$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}.$$

b) Since $\nabla \cdot \vec{B} = 0$ everywhere, $\vec{B}$ is solenoidal and thus there exists a vector field $\vec{A}$ such that $\nabla \times \vec{A} = \vec{B}$.

Now, $$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = \nabla \times \vec{E} + \frac{\partial}{\partial t} \nabla \times \vec{A}$$
$$= \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B}$$
$$= \vec{0},$$

so $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ is conservative.

c) $$\iint_\Sigma \vec{B} \cdot \hat{n} \, dA = \iint_\Sigma (\nabla \times \vec{A}) \cdot \hat{n} \, dA$$
$$= \oint_{\partial \Sigma} \vec{A} \cdot d\vec{r}$$

7) a) $f(z) = \frac{z^2 \cosh z}{(z-10)^3}$ has singularity only at $z = 10$, which is not inside Γ.

Hence $\oint_\Gamma \frac{z^2 \cosh z}{(z-10)^3} \, dz = \boxed{0}$ by <u>Cauchy-Goursat Theorem</u>.

b) $f(z) = z^4 \sin\left(\frac{1}{jz}\right)$ has a singularity at $z = 0$, which <u>is</u> inside Γ.

Expanding out the Laurent series of f at $z = 0$,

$$f(z) = z^4 \left(\frac{1}{jz} - \frac{1}{3!} \frac{1}{(jz)^3} + \frac{1}{5!} \frac{1}{(jz)^5} - \frac{1}{7!} \frac{1}{(jz)^7} \cdots \right)$$
$$= -j z^3 - j \frac{1}{3!} z - \underline{j \frac{1}{5!}} \frac{1}{z} - j \frac{1}{7!} \frac{1}{z^3} - \cdots$$

So $\mathrm{Res}(f, 0) = -j \frac{1}{5!}$.

By the <u>Residue theorem</u>,
$$\oint_\Gamma f(z) \, dz = 2\pi j \, \mathrm{Res}(f, 0) = 2\pi j \left(-j \frac{1}{5!}\right) = \frac{\pi}{3 \cdot 4 \cdot 5} = \boxed{\frac{\pi}{60}}$$

c) $(z^2+4)^2 = (z-2j)^2(z+2j)^2$, so we can write $f(z)$ as

$$f(z) = \frac{e^{jz}}{(z^2+4)^2} = \frac{e^{jz}/(z+2j)^2}{(z-2j)^2}.$$

By Cauchy's Integral formula,

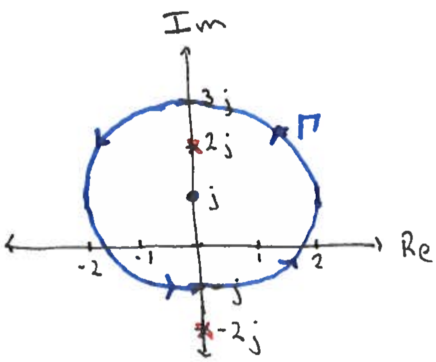
$$\oint_\Gamma f(z)\,dz = 2\pi j \left[\frac{d}{dz}\left(\frac{e^{jz}}{(z+2j)^2}\right)\right]_{z=2j}$$

$$= 2\pi j\left[\frac{e^{jz}}{(z+2j)^3}(jz-4)\right]_{z=2j}$$

$$= 2\pi j\left(\frac{e^{-2}}{(4j)^3}(-2-4)\right)$$

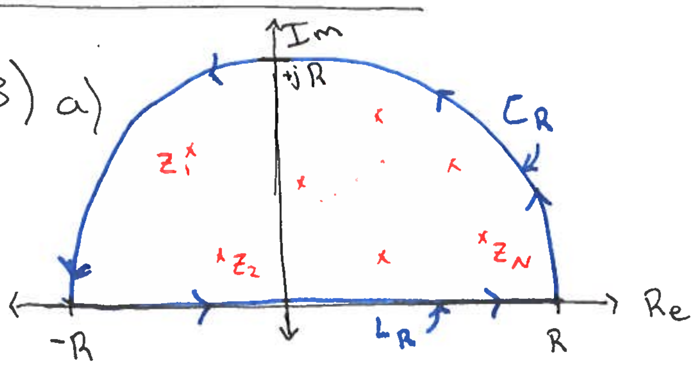
$$= \frac{-6j\,e^{-2}}{4^3} = \frac{-3j\,e^{-2}}{2^5} = -j\,\frac{3e^{-2}}{32}$$

$$= \boxed{-j\,\frac{3}{32e^2}}.$$



f has singularities at $z = \pm 2j$.

8) a)



Suppose f is analytic except at finitely many points $z_1, \ldots, z_N$ in the upper half plane, and f is analytic on the real line.

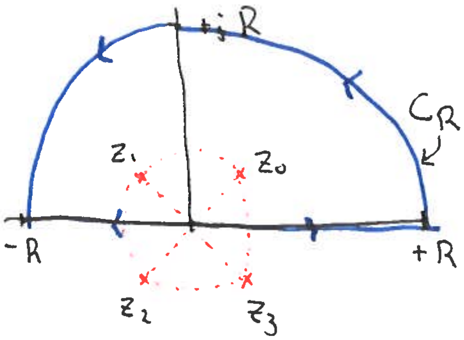
Let C_R denote the semicircular path of radius R centered at the origin, going counterclockwise from R to $-R$ in the upper half plane. Suppose further that

$$\lim_{R\to\infty}\left[\left(\max_{z\in C_R}|f(z)|\right)\pi R\right] = 0.$$

Then $$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi j \sum_{k=1}^{N} \operatorname{Res}(f, z_k).$$

b) Let $f(z) = \frac{1}{1+z^4}$, which has singularities at all of the 4^{th} roots of -1. These are $z_k = e^{j\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)} = e^{j\pi\left(\frac{1}{4}+\frac{k}{2}\right)}$ for $k = 0, 1, 2, 3$, or

$$z_0 = e^{j\frac{\pi}{4}}, \quad z_1 = e^{j\frac{3\pi}{4}}, \quad z_2 = e^{j\frac{5\pi}{4}}, \quad z_3 = e^{j\frac{7\pi}{4}}.$$



Only z_0 and z_1 lie in the upper half plane.

The residues of f can be computed as

$$\mathrm{Res}(f, z_k) = \frac{1}{4 z_k^3},$$

and thus

$$\mathrm{Res}(f, z_0) = \frac{1}{4e^{j3\pi/4}} = \frac{1}{4}e^{-j3\pi/4} = -\frac{1}{4\sqrt{2}}(1+j)$$

$$\mathrm{Res}(f, z_1) = \frac{1}{4e^{j9\pi/4}} = \frac{1}{4}e^{-j\frac{9\pi}{4}} = \frac{1}{4}e^{-j\frac{\pi}{4}} = \frac{1}{4\sqrt{2}}(1-j).$$

Since $|f(z)| = \frac{1}{|1+z^4|} \leq \frac{1}{|z|^4 - 1} = \frac{1}{R^4-1}$ for $z = Re^{j\theta}$ on C_R,

by the ML-theorem $\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R}{R^4-1} \to 0$ as $R \to \infty$,

So

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi j\left(\mathrm{Res}(f, z_0) + \mathrm{Res}(f, z_1)\right)$$

$$= \frac{2\pi j}{4\sqrt{2}}\left(-(\cancel{1}+j) + \cancel{1} - j\right) = \boxed{\frac{\pi}{\sqrt{2}}}$$

9) a) $\nabla \times (f(r)\,\vec{r}) = (\nabla f(r)) \times \vec{r} + f(r)\,(\underbrace{\nabla \times \vec{r}}_{\vec{0}})$

$$= \frac{f'(r)}{r}\,\underbrace{\vec{r} \times \vec{r}}_{\vec{0}} + \vec{0} = \vec{0}$$

Since $\nabla f(r) = \hat{\imath}\,\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \hat{\jmath}\,\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}$

$$= \hat{\imath}\, f'(r)\,\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{\jmath}\, f'(r)\,\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{k}\, f'(r)\,\frac{z}{\sqrt{z^2+y^2+x^2}}$$

$$= \frac{f'(r)}{r}\,(x, y, z) = \frac{f'(r)}{r}\,\vec{r},$$

and $\nabla \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \vec{0}$.

b) $\oint_{\Gamma_1} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \underbrace{(\vec{\nabla} \times \vec{F})}_{\vec{0}} \cdot \hat{n}\, dA = 0$.

c) $\Phi(x,y,z) = g(r)$

$$\nabla g = \frac{g'(r)}{r}\,\vec{r} = f(r)\,\vec{r}$$

$$\Rightarrow \boxed{f(r) = \frac{g'(r)}{r}} \quad \text{or} \quad \boxed{g'(r) = r f(r)}$$

d) $f(r) = -\frac{1}{r^3}$ $\qquad g'(r) = r\left(-\frac{1}{r^3}\right) = -\frac{1}{r^2}$

$$\Rightarrow g(r) = \frac{1}{r} + c \quad \text{for some constant } c \in \mathbb{R}.$$

For $\vec{r} = (-1, -2, -3)$ and $\vec{r} = (1, 2, 3)$, $r = \sqrt{1^2+2^2+3^2} = \sqrt{14}$.

Both points are the same distance from the origin, So $\Psi(-1,-2,-3) = \Psi(1,2,3)$ and thus

$$\int_{\Gamma_2} -\frac{\vec{r}}{r^3} \cdot d\vec{r} = \Psi(-1,-2,-3) - \Psi(1,2,3)$$
$$= 0.$$

10) a) $$\iint_{\partial\Omega} \vec{C} \cdot \hat{n}\, ds = \iiint_{\Omega} \underbrace{(\nabla \cdot \vec{C})}_{0} dV$$
$$= 0$$

by the divergence theorem.

b) $$0 = \iint_{\partial\Omega} (f \nabla f) \cdot \hat{n}\, dA$$
$$= \iiint_{\Omega} (\nabla \cdot (f \nabla f))\, dV \quad \text{(by divergence theorem)}$$
$$= \iiint_{\Omega} (\nabla f \cdot \nabla f + f \nabla^2 f)\, dV$$
$$= \iiint_{\Omega} (\|\nabla f\|^2 + f \underbrace{\nabla^2 f}_{0})\, dV$$

$\nabla^2 f = 0$ by assuptic

$$= \iiint_{\Omega} \|\nabla f\|^2 dV.$$

Since $\iiint_{\Omega} \|\nabla f\|^2 dV = 0$ and $\|\nabla f\|^2 \geq 0$,

this implies that $\|\nabla f\|^2 = 0$ everywhere in Ω.

Hence $\nabla f = 0$ and thus f is constant on Ω.

But $f = 0$ on $\partial\Omega$, so $f = 0$ on all of Ω. ■