ECE 206 – University of Waterloo	**Fall 2019**
Lecture notes for Week 0	
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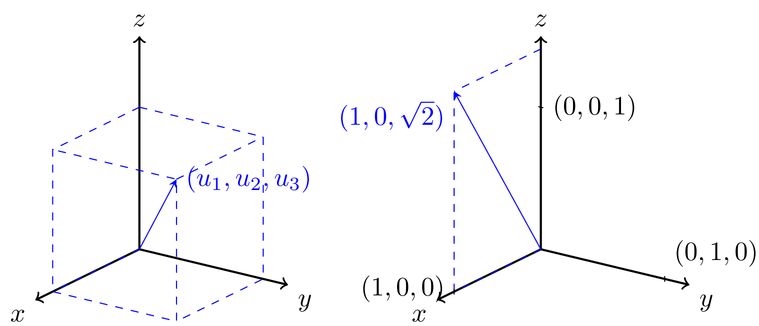
These notes provide a quick refresher on vectors, particularly in three-dimensional space. This won't be explicitly covered in class, but you will be expected to know the material here.

0.1 Notation for vectors

Definition 0.1. A (real) *n-dimensional vector* $\boldsymbol{v}$ is an n-tuple $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ of real numbers $v_1, \ldots, v_n \in \mathbb{R}$. The collection of all real n-dimensional vectors is denoted as $\mathbb{R}^n$.

On paper/chalkboard, we typically write $\vec{v}$ instead of $\boldsymbol{v}$, since boldface is difficult to indicate in handwriting. In this course, we are typically only concerned with vectors in $\mathbb{R}^2$ (on the plane) and $\mathbb{R}^3$ (in space). There are a few different equivalent ways that we will use to represent vectors in this course. In three dimensions, we have:

- Visualization as arrows (starting at the origin) in three-dimensional space:



- Ordered lists of numbers

$$\boldsymbol{u} = (u_1, u_2, u_3) \qquad (x, y, z), \qquad (1, 0, \sqrt{2}), \qquad \text{etc.}$$

- Column vectors

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix}, \qquad \text{etc.}$$

- ijk-notation

$$\boldsymbol{u} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_1\hat{\boldsymbol{k}}, \qquad x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{k}}, \qquad \text{etc.}$$

  where $\hat{\boldsymbol{\imath}} = (1, 0, 0)$, $\hat{\boldsymbol{\jmath}} = (0, 1, 0)$, and $\hat{\boldsymbol{k}} = (0, 0, 1)$ are the *standard basis unit vectors*.

One important vector that we must distinguish in $\mathbb{R}^3$ is the *zero vector* (or the *the origin*), which is denoted $\mathbf{0} = (0, 0, 0)$ (use $\vec{0}$ when writing this on paper/blackboard). Analogous properties hold for two-dimensional vectors. In two-dimensions, we use the standard unit vectors

$$\hat{\boldsymbol{\imath}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\boldsymbol{\jmath}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From here on out, we'll only focus on three-dimensional vectors in these notes.

The important defining features of $\mathbb{R}^3$ as a *vector space* are scalar multiplication and vector addition.

Properties. For any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ and any scalar $a \in \mathbb{R}$, we have

1. *scalar multiplication*: $a\boldsymbol{u} = (au_1, au_2, au_3)$
2. *vector addition*: $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

0.2 Dot product, norm, and angle

Definition 0.2. The *dot product* of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

With column notation, we can write this as the matrix product

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^\intercal \boldsymbol{v} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Note that the result of the dot product is a number.

Definition 0.3. The *norm* of a vector $\boldsymbol{v} \in \mathbb{R}^3$ is defined as

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + v_3^3}.$$

Properties. The norm satisfies the following properties:

1. $\|a\boldsymbol{v}\| = |a|\|\boldsymbol{v}\|$ for all $a \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^2$.
2. $\|\boldsymbol{v}\| \geq 0$ for all $\boldsymbol{v} \in \mathbb{R}^3$ (and $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$).
3. $\|\boldsymbol{v} + \boldsymbol{u}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{u}\|$ for all $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^3$. (This is called the *triangle inequality*).
4. $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ for all $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^3$. (This is called the *Cauchy-Schwarz inequality*).

Definition 0.4. The *angle* between two nonzero vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ is defined as the unique number $\theta \in [0, \pi]$ that satisfies

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta.$$

That is, $\theta = \arccos\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$.

Example 0.5. Compute the angle between the vectors.

1. Find the angle between $\boldsymbol{u} = (2, 1, -2)$ and $\boldsymbol{v} = (1, -7, 20)$.

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= (2)(1) + (1)(-7) + (-2)(20) = 2 - 7 - 40 = -45 \\ \|\boldsymbol{u}\| &= \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3 \\ \|\boldsymbol{v}\| &= \sqrt{1^2 + (-7)^2 + (20)^2} = \sqrt{1 + 49 + 200} = \sqrt{450} = 15\sqrt{2} \end{aligned}$$

   One therefore has that

$$\theta = \arccos\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right) = \arccos\left(\frac{-45}{45\sqrt{2}}\right) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} \qquad (= 135^\circ).$$

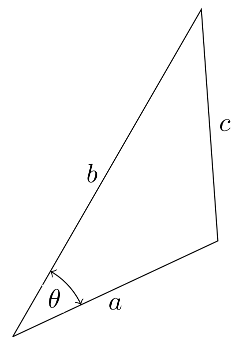
2. Find the angle between $\boldsymbol{u} = (1, 2, -1)$ and $\boldsymbol{v} = (3, -1, 1)$.

$$\boldsymbol{u} \cdot \boldsymbol{v} = 3 + (-2) + (-1) = 0$$

   One therefore has that $\cos\theta = 0$ and thus $\theta = \pi/2$.

Definition 0.6. Two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ are said to be *orthogonal* if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. We often write $\boldsymbol{u} \perp \boldsymbol{v}$ to denote when $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

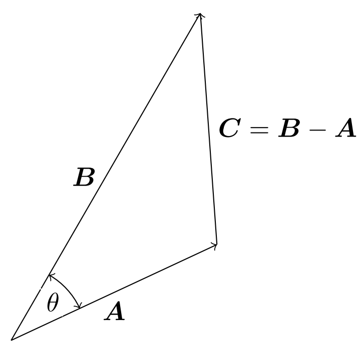
Problem. Prove the law of cosines: If a, b, c are the lengths of the sides of any triangle,



where θ is the angle between the sides of length a and b, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Proof. Let the origin be the point joining the segments of length a and b, and define the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ as the sides of the triangle. Define the vector $\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{A}$ as in the diagram:



One has:

$$\begin{aligned} c^2 = \|\boldsymbol{C}\|^2 = \boldsymbol{C} \cdot \boldsymbol{C} &= (\boldsymbol{B} - \boldsymbol{A}) \cdot (\boldsymbol{B} - \boldsymbol{A}) \\ &= \boldsymbol{B} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{A} - \boldsymbol{B} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{B} \\ &= \|\boldsymbol{B}\|^2 + \|\boldsymbol{A}\| - 2(\boldsymbol{A} \cdot \boldsymbol{B}) \\ &= \|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\| - 2\|\boldsymbol{A}\|\|\boldsymbol{B}\|\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta, \end{aligned}$$

as desired. □

0.3 Cross product

Definition 0.7. The *cross product* of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$ is the vector $\boldsymbol{u} \times \boldsymbol{v}$ defined as

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2 v_3 - u_3 v_2)\hat{\boldsymbol{\imath}} + (u_3 v_1 - u_1 v_3)\hat{\boldsymbol{\jmath}} + (u_1 v_2 - u_2 v_1)\hat{\boldsymbol{k}}.$$

Note that the cross product results in a vector.

A neat way to compute the cross product of two vectors is to use a determinant:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\boldsymbol{\imath}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\boldsymbol{\jmath}} + \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\boldsymbol{k}}.$$

Note the minus sign in the determinant before the $\hat{\boldsymbol{\jmath}}$ part! Don't forget it!

Properties. The cross product has the following properties.

1. $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$ for all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$
2. $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$ for all vectors $\boldsymbol{u} \in \mathbb{R}^3$
3. The vector $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.
4. $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

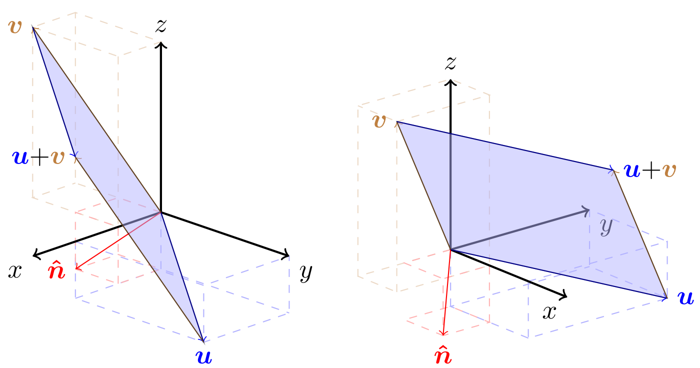
Any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$ define a *parallelogram* (see the example below). An important fact is that the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ can be used to compute the area of the resulting parallelogram. Indeed, the resulting area is

$$\text{Area} = \|\boldsymbol{u} \times \boldsymbol{v}\|.$$

Meanwhile, the *normal unit vector* that is perpendicular to the plane of of the parallelogram is

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{u} \times \boldsymbol{v}}{\|\boldsymbol{u} \times \boldsymbol{v}\|}.$$

Example 0.8. Consider the vectors $\boldsymbol{u} = (2, 3, -1)$ and $\boldsymbol{v} = (1, -2, 3)$. Compute the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.



The area of the parallelogram is $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$, there θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. We have

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{vmatrix} \\ &= ((3)(3) - (-2)(-1))\hat{\boldsymbol{\imath}} + ((3)(2) - (1)(-1))\hat{\boldsymbol{\jmath}} + ((2)(-2) - (3)(2))\hat{\boldsymbol{k}} \\ &= (7, -7, -7) \\ &= 7(1, -1, -1) \end{aligned}$$

and thus the area of the parallelogram is

$$\text{Area} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta = \|\boldsymbol{u} \times \boldsymbol{v}\| = 7\|(1, -1, -1)\| = 7\sqrt{3}.$$

Alternatively, we may note that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ and that

$$\boldsymbol{u}\cdot\boldsymbol{v} = 2-6-3 = -7.$$

The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is computed as $\theta = \arccos\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ and $\sin\theta = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. Hence

$$\text{Area} = \|\boldsymbol{u}\times\boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta = 7\sqrt{3}.$$

The normal unit vector is $\hat{\boldsymbol{n}} = (1,-1,-1)/\sqrt{3}$. To verify that this vector is indeed orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$, note that

$$\begin{aligned} & (1,-1,-1)\cdot\boldsymbol{u} = (1,-1,-1)\cdot(2,3,-1) = 2-3+1 = 0 \\ \text{and} \qquad & (1,-1,-1)\cdot\boldsymbol{v} = (1,-1,-1)\cdot(1,-2,3) = 1+2-3 = 0. \end{aligned}$$