ECE 206 – University of Waterloo **Fall 2019**

Lecture notes for Week 2

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2.1 Line integrals of scalar and vector fields

Roughly speaking, a *field* defines how a scalar-valued or a vector-valued quantity varies in space.

2.1.1 Scalar fields

Definition 2.1 (***Scalar fields***)**.** Let $D \subset \mathbb{R}^n$ be a region of space. A *scalar field* on D is a function $f : D \to \mathbb{R}$. The region D is the *domain* of f.

An arbitrary point in $\mathbb{R}^n$ is typically denoted as $\boldsymbol{r}$, with components denoted by

$$\boldsymbol{r} = (x_1, x_2, \ldots, x_n).$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the standard names of the components of the variable $\boldsymbol{r}$ are $\boldsymbol{r} = (x, y)$ and $\boldsymbol{r} = (x, y, z)$. A field typically represents some sort of physical quantity that varies in space. Some examples of physical quantities that are typically represented by fields include:

- Temperature, atmospheric pressure, and humidity of a point in space
- Density (such as mass density or charge density)
- Potential fields (such as gravitational or electrostatic potential)

Example 2.2. Suppose the temperature at any given point on a square plate, which extends over $0 \leq x \leq 1$ and $0 \leq y \leq 1$, is given by

$$T(x, y) = \frac{100}{(x+1)^2 + (y+1)^2}.$$

The temperature T is a scalar field whose domain is the two-dimensional square $D = \{(x, y) \,|\, x, y \in [0, 1]\}$. A scalar field in $\mathbb{R}^2$ can be visualized by examining its *graph* in $\mathbb{R}^3$, or by viewing its *level curves* in $\mathbb{R}^2$. The temperature field defined in this example is depicted in Figure 2.1.

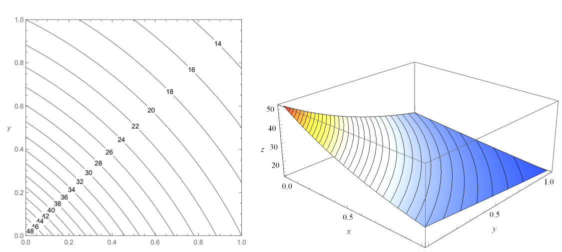


Figure 2.1: Visualizing the temperature field in Example 2.2 from its level curves and its graph.

2.1.2 Line integrals of scalar fields

Let $f : D \to \mathbb{R}$ be a scalar field on some region $D \subset \mathbb{R}^n$ of space and let $\boldsymbol{\gamma} : [a, b] \to D$ be a C^1 path. The *path integral* (or *line integral*) *along the path* $\boldsymbol{\gamma}$ is equal to

$$\int_a^b f(\boldsymbol{\gamma}(t)) \, \|\boldsymbol{\gamma}'(t)\| \, dt.$$

If $\boldsymbol{\gamma}$ is only *piecewise* C^1, then we may split the integral over subintervals $[a_0, a_1]$, $[a_1, a_2]$, $\ldots$, $[a_{N-1}, a_N]$ such that $\boldsymbol{\gamma}$ is C^1 and regular on each subinterval and compute the path integral by summing up the integrals over the subintervals

$$\int_a^b f(\boldsymbol{\gamma}(t)) \, \|\boldsymbol{\gamma}'(t)\| \, dt = \int_{a_0}^{a_1} f(\boldsymbol{\gamma}(t)) \, \|\boldsymbol{\gamma}'(t)\| \, dt + \cdots + \int_{a_{N-1}}^{a_N} f(\boldsymbol{\gamma}(t)) \, \|\boldsymbol{\gamma}'(t)\| \, dt.$$

Recalling from last week our formalism for computing the distance travelled by a path, we see that the integral computing the distance is just a path integral where we take the constant function $f(\boldsymbol{r}) = 1$ for all $\boldsymbol{r}$.

Now suppose that $\Gamma \subset D$ is a C^1 simple curve in $\mathbb{R}^n$. Analogous to our analysis from last time, where we showed that the length of the curve is independent of parameterization, it can also be shown that path integrals are also independent of parameterization. The *path integral* along the simple C^1 curve Γ is denoted

$$\int_\Gamma f \, ds. \tag{2.1}$$

If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ is any regular C^1 parameterization of Γ, then the value of this path integral can be computed as

$$\boxed{\int_\Gamma f \, ds = \int_a^b f(\boldsymbol{\gamma}(t)) \, \|\boldsymbol{\gamma}'(t)\| \, dt}$$

and this is independent of the parameterizing path.

The value of a path integral of a scalar field along a curve can be thought of as the area under the curve along the surface of the graph of the scalar field, as shown in Figure 2.2

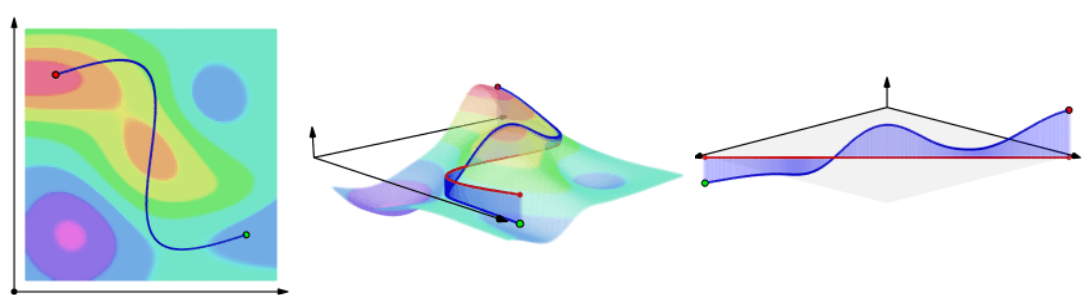


Figure 2.2: A path integral of a scalar field can be thought of as the area under the curve along the surface of the graph of the scalar field.

Example 2.3. Suppose a helical wire in space is modeled by the helix that is parameterized by the path $\boldsymbol{\gamma} : [0, 4\pi] \to \mathbb{R}^3$ defined as

$$\boldsymbol{\gamma}(t) = (R \cos t, R \sin t, 3t)$$

where $R > 0$ is the radius of the helix. If the linear charge density of the wire at a point $\boldsymbol{r} = (x, y, z)$ is given by $\rho(x, y, z) = xyz$, what is the total charge of the wire?

Here we must evaluate the path integral

$$\int_\Gamma \rho\, ds$$

using the parameterization defined above. Note that the speed of the path is

$$\gamma(t) = \sqrt{R^2(\sin^2 t + \cos^2 t) + 3} = \sqrt{R^2 + 9},$$

and the charge density of a point of the wire wire along the path is

$$\rho(\gamma(t)) = 3t \cos t \sin t.$$

The total charge of the wire is therefore

$$\begin{aligned} \int_\Gamma \rho\, ds &= \int_0^{4\pi} 3t \cos t \sin t \sqrt{R^2+9}\, dt \\ &= 3\sqrt{R^2+9} \int_0^{4\pi} \frac{1}{2} t \sin 2t\, dt \\ &= -3\pi\sqrt{R^2+9}. \end{aligned}$$

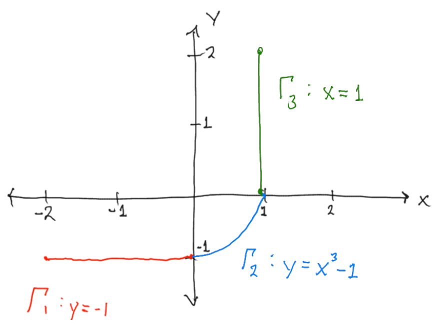
Remark. If a simple curve $\Gamma \subset \mathbb{R}^n$ is only piecewise C^1, it may be thought of as the union of a bunch of C^1 curves,

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N,$$

and the path integral along this piecewise C^1 curve is defined as

$$\int_\Gamma f\, ds = \int_{\Gamma_1} f\, ds + \int_{\Gamma_2} f\, ds + \cdots \int_{\Gamma_N} f\, ds.$$

Example 2.4. Let Γ be the piecewise C^1 curve that is the union of the curves below. Evaluate $\displaystyle\int_\Gamma x^3\, ds$.



We must first parameterize each of the component curves. We may choose

$$\begin{aligned} &\Gamma_1: \quad \gamma_1(t) = (t, -1) &&\text{for } -2 \le t \le 0 \\ &\Gamma_2: \quad \gamma_2(t) = (t, t^3 - 1) &&\text{for } 0 \le t \le 1 \\ &\Gamma_3: \quad \gamma_3(t) = (1, t) &&\text{for } 0 \le t \le 2 \end{aligned}$$

The line integrals over each of these curves are

$$\begin{aligned} \int_{\Gamma_1} x^3\, ds &= \int_{-2}^0 t^3 \sqrt{(1)^2 + (0)^2}\, dt = \int_{-2}^0 t^3\, dt = -4 \\ \int_{\Gamma_2} x^3\, ds &= \int_0^1 t^3 \sqrt{(1)^2 + (3t^2)^2}\, dt = \int_0^1 t^3\sqrt{1+9t^4}\, dt = \frac{1}{54}\left(10^{3/2} - 1\right) \approx 0.57 \\ \int_{\Gamma_3} x^3\, ds &= \int_0^2 1^3 \sqrt{(0)^2 + (1)^2}\, dt = \int_0^2 1\, dt = 2. \end{aligned}$$

The value of the line integral that we were asked to compute is

$$\int_{\Gamma_1} x^3\, ds + \int_{\Gamma_1} x^3\, ds + \int_{\Gamma_1} x^3\, ds \approx -1.43.$$

Finally, we must introduce the idea of differentiability for scalar fields.

Definition 2.5 (***Differentiability for scalar fields***)**.** Let $f : D \to \mathbb{R}$ be a scalar field on a region $D \subset \mathbb{R}^n$.

- We say that f is C^1 if all of the partial derivatives

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$$

  exist and are continuous on all of D.

- We say that f is C^2 if all of the partial derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

  exist and are continuous on D for all indices $i, j \in \{1, \dots, n\}$.

2.1.3 Vector fields

Definition 2.6 (***Vector field***)**.** A *vector field* on a region $D \subset \mathbb{R}^n$ is a function $\boldsymbol{F} : D \to \mathbb{R}^n$.

A vector field is a *vector-valued* function that assigns a vector $\boldsymbol{F}(\boldsymbol{r}) \in \mathbb{R}^n$ to each point $\boldsymbol{r} \in D$. Similar to paths, a vector field is typically defined in terms of its *component functions*, and one writes

$$\boldsymbol{F}(\boldsymbol{r}) = \big(F_1(\boldsymbol{r}), F_2(\boldsymbol{r}), \dots, F_n(\boldsymbol{r})\big),$$

where each of the components $F_1, \dots, F_n$ is a scalar field.

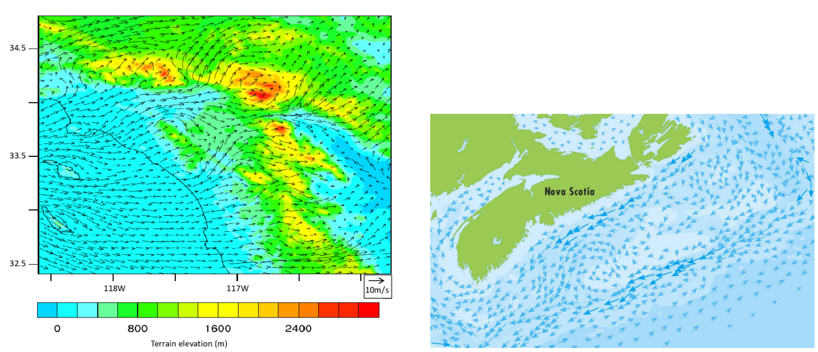
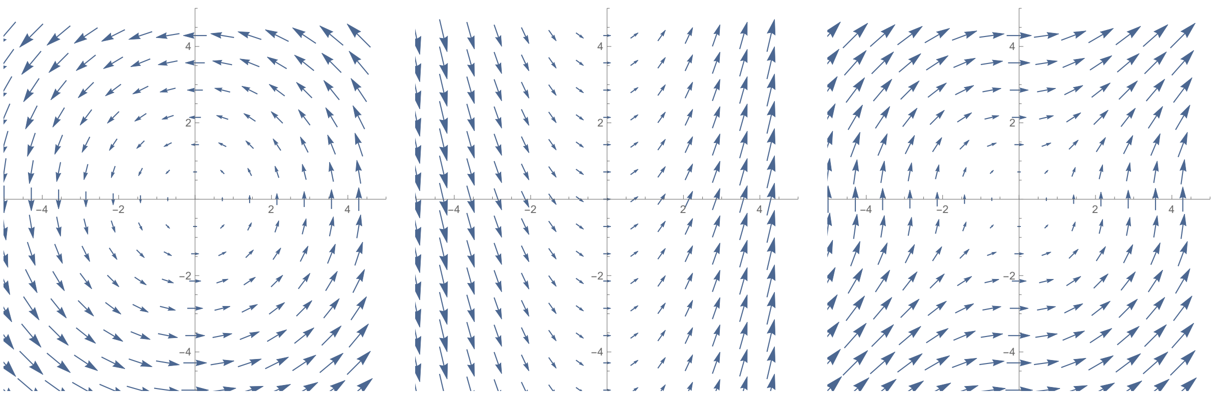


Figure 2.3: Some physical examples of vector fields.

Vector fields are usually only defined in two- or three-dimensional space and can be used to represent physical vector-valued quantities at each point in space. Some physical examples of vector fields include:

- Wind speed and direction for weather maps.
- Ocean currents or flow speed of a fluid in a pipe.
- Force fields (e.g. gravitational, electric, and magnetic fields) that indicate the force felt by a particle at that position.

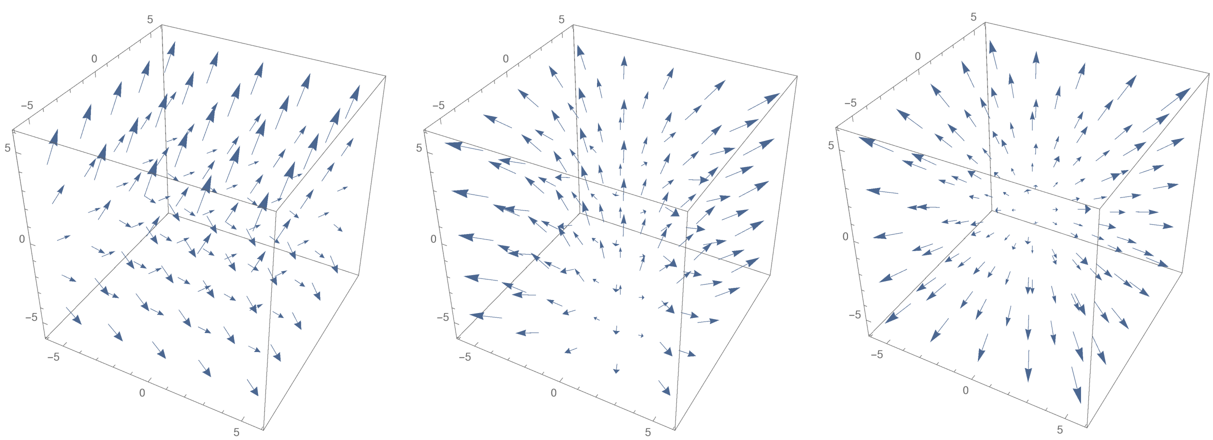
A vector field $\boldsymbol{F}$ in $\mathbb{R}^2$ can be visualized by drawing the a scaled version of the vector $\boldsymbol{F}(x, y)$ at various points (x, y) on the plane. Some examples of vector fields are shown in Figure 2.4.



(a) $\boldsymbol{F}(x,y) = -y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}$ (b) $\boldsymbol{F}(x,y) = (1,\, x)$ (c) $\boldsymbol{F}(x,y) = \big(\ln(1+y^2), \ln(1+x^2)\big)$

Figure 2.4: Some examples of vector fields in $\mathbb{R}^2$.

Vector fields in $\mathbb{R}^3$ can be visualized in a similar manner.



(a) $\boldsymbol{F}(x,y,z) = (1, 2, z)$ (b) $\boldsymbol{F}(x,y,z) = (x, y, 3)$ (c) $\boldsymbol{F}(x,y,z) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$

Figure 2.5: Some examples of vector fields in $\mathbb{R}^3$.

Example 2.7. Suppose a point particle with charge q is located at the origin. The *electric field* resulting from this point charge is given by

$$\boldsymbol{E}(x,y,z) = \left(\frac{kqx}{(x^2+y^2+z^2)^{3/2}}, \frac{kqy}{(x^2+y^2+z^2)^{3/2}}, \frac{kqz}{(x^2+y^2+z^2)^{3/2}} \right),$$

where $k \approx 9.0 \times 10^9 \,\mathrm{N \cdot m^2/C^2}$ is the *electrostatic constant* (or *Coulomb's constant*). We can also write this field using $\boldsymbol{r} = (x, y, z)$ as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{kq\boldsymbol{r}}{r^3}$$

where r is the magnitude of $\boldsymbol{r}$

$$r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}.$$

Note that this vector field is clearly not defined at the origin $\boldsymbol{r} = \boldsymbol{0}$, since $r = \|\boldsymbol{r}\| = 0$ at this point and we cannot divide by zero. The domain of this vector field is therefore everywhere in $\mathbb{R}^3$ except the origin:

$$D = \{\boldsymbol{r} \in \mathbb{R}^3 \,|\, \boldsymbol{r} \neq \boldsymbol{0}\}$$

which we can also write as

$$D = \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$$

(i.e., all of $\mathbb{R}^3$ with the origin removed). This vector field can be visualized as in the following figure. Notice how the magnitude drops off significantly away from the origin.

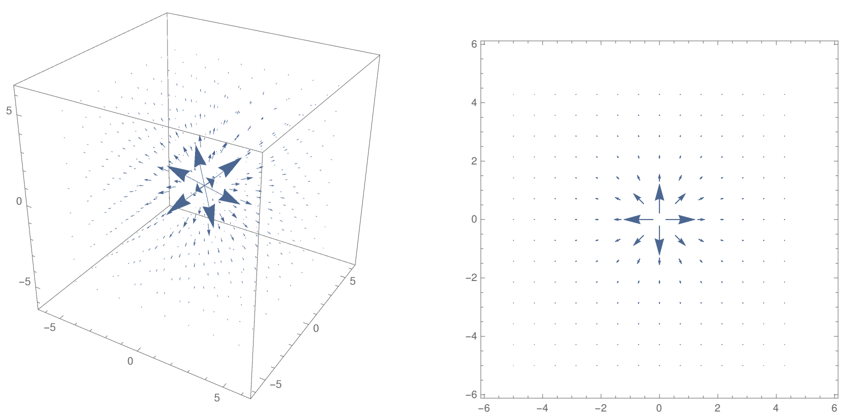


Figure 2.6: Visualization of the electric field emanating from a point charge at the origin.

The vector field in Example 2.7 is an example of a *radial* vector field, since field $\boldsymbol{F}(\boldsymbol{r})$ at a point $\boldsymbol{r}$ always points in the same direction as $\boldsymbol{r}$ and the magnitude of $\boldsymbol{F}(\boldsymbol{r})$ depends only on $r = \|\boldsymbol{r}\|$, the magnitude of $\boldsymbol{r}$. Redial vector fields are of the form

$$\boldsymbol{F}(\boldsymbol{r}) = f(r)\boldsymbol{r}$$

for some function f, where $\boldsymbol{r} = (x_1, x_2, \ldots, x_n)$ and

$$r = \|\boldsymbol{r}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

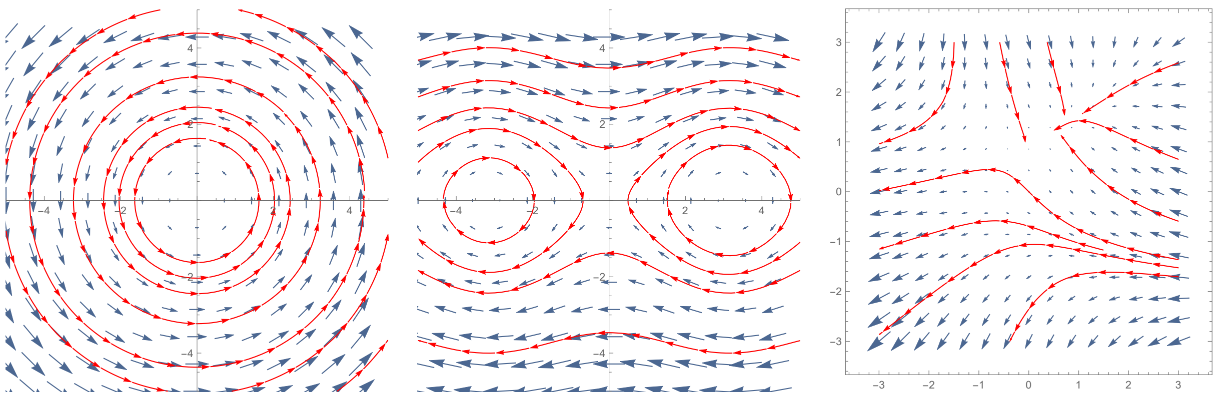
We also need a notion of differentiability for vector fields.

Definition 2.8 (***Differentiability of vector fields***)**.** Let $\boldsymbol{F} : D \to \mathbb{R}$ be a vector field on a region $D \subset \mathbb{R}^n$ with component functions $\boldsymbol{F}(\boldsymbol{r}) = \big(F_1(\boldsymbol{r}), F_2(\boldsymbol{r}), \ldots, F_n(\boldsymbol{r})\big)$. For any integer k, the vector field $\boldsymbol{F}$ is C^k if each of the component functions $F_1, \ldots, F_n$ are C^k as scalar fields.

2.1.3.1 Flow lines

Visualizing a vector field by drawing many different vectors at each point can be rather cumbersome. Another way of visualizing a vector field is by drawing some of its *flow lines*. The flow lines give an intuition of the 'flow' of the vector field. Mathematical software can be used to draw flow lines using a computer.

> **Definition 2.9** (***Flow lines***)**.** Let $\boldsymbol{F} : D \to \mathbb{R}^n$ be a vector field in $\mathbb{R}^n$. A *flow line* (or a *streamline*) of $\boldsymbol{F}$ is a path $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ such that $\boldsymbol{\gamma}'(t) = \boldsymbol{F}(\boldsymbol{\gamma}(t))$ for all t.



(a) $\boldsymbol{F}(x, y) = -y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}$ (b) $\boldsymbol{F}(x, y) = (y, \sin x)$ (c) $\boldsymbol{F}(x, y) = \left(-1 - x^2,\ 1 + x - y^2\right)$

Figure 2.7: Some vector fields in $\mathbb{R}^2$ with some of their flow lines drawn.

Finding the equation for a path of a flow line amounts to solving a system of differential equations. For example, in $\mathbb{R}^2$, a path and a vector field are given by their coordinate functions

$$\boldsymbol{\gamma}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} \qquad \text{and} \qquad \boldsymbol{F}(x, y) = F_1(x, y)\,\hat{\boldsymbol{\imath}} + F_2(x, y)\,\hat{\boldsymbol{\jmath}}.$$

Equating the components of the velocity $\boldsymbol{\gamma}'(t)$ of the path and the vector field $\boldsymbol{F}(\boldsymbol{\gamma}(t))$ yields the system

$$\begin{cases} x'(t) = F_1(x(t), y(t)) \\ y'(t) = F_2(x(t), y(t)). \end{cases}$$

Finding a pair of functions $x(t)$ and $y(t)$ that simultaneously satisfy these equations gives us a flow line.

Example 2.10. Here are some examples of fields in $\mathbb{R}^2$ for which we can solve the differential equations to find equations for their flow lines.

1. For the vector field $\boldsymbol{F}(x, y) = (-y, x)$, the differential equations for the flow lines are

$$\frac{dx}{dt} = -y \qquad \text{and} \qquad \frac{dy}{dt} = x.$$

One way to find solutions is to multiply the first equation by y and the second equation by x to get

$$x\frac{dx}{dt} = -xy \qquad \text{and} \qquad y\frac{dy}{dt} = xy.$$

Now we can equate

$$y\frac{dy}{dt} = -x\frac{dx}{dt} \qquad \text{to find} \qquad x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad \Longrightarrow \qquad \frac{1}{2}\frac{d}{dt}\left(x^2 + y^2\right) = 0.$$

That is, the variables $x(t)$ and $y(t)$ satisfy the equations of a circle:

$$x^2 + y^2 = c$$

for some constant c. Different values of c yield different curves. This vector field with some of its flow lines drawn can be seen in Figure 2.7(a).

2. For the vector field $\boldsymbol{F}(x, y) = (1, x)$, the differential equations defining the flow lines are

$$\frac{dx}{dt} = 1 \qquad \text{and} \qquad \frac{dy}{dt} = x.$$

We may take $x = t$ and solve the differential equation

$$\frac{dy}{dt} = t$$

which has solutions

$$y(t) = \frac{1}{2}t^2 + c$$

for a constant c. The resulting paths trace out parabolas.

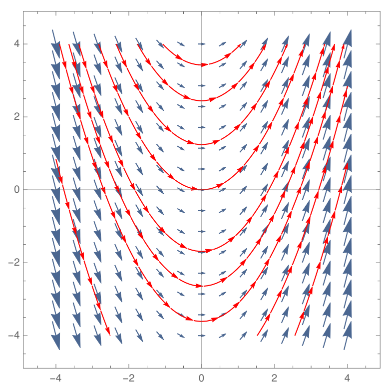


Figure 2.8: Flow lines for the field $\boldsymbol{F}(x, y) = (1, x)$ are graphs of parabolas $y = \frac{1}{2}x^2 + c$.

3. For the vector field $\boldsymbol{F}(x, y) = (1, y)$, the differential equations defining the flow lines are

$$\frac{dx}{dt} = 1 \qquad \text{and} \qquad \frac{dy}{dt} = y.$$

We may similarly take $x(t) = t$. To solve the second differential equation,

$$\frac{dy}{dt} = y \qquad \Longrightarrow \qquad \int \frac{1}{y}\, dy = \int dt \qquad \Longrightarrow \qquad \ln|y| = t + c_0$$

for a constant c_0. Rearranging yields

$$y = \pm e^t e^{c_0}$$

so the solutions are $y(t) = ce^t$ for a constant c. The resulting flow lines are graphs of exponential functions.

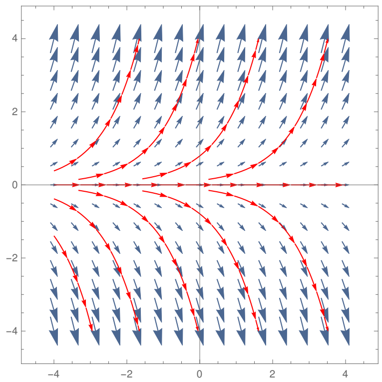


Figure 2.9: Flow lines for the field $\boldsymbol{F}(x,y) = (1, y)$ are graphs of exponential functions $y = ce^x$.

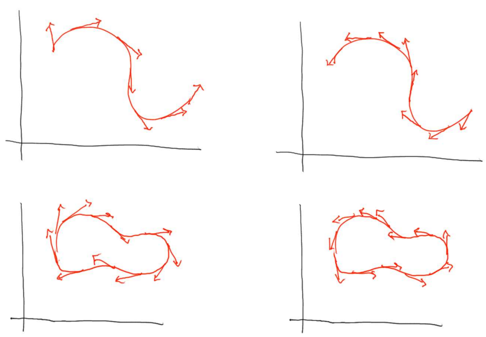


Figure 2.10: A simple C^1 curve can have two different orientations.

2.1.4 Oriented curves

An *orientation* of a C^1 curve is a way of continuously assigning a unit vector to each point of the curve that is tangent to the curve at that point. (Recall that a vector $\boldsymbol{v}$ is a *unit vector* if $\|\boldsymbol{u}\| = 1$.) Intuitively, we see that any simple curve can have exactly two different possible orientations.

Any simple C^1 curve (or simple piecewise C^1 curve) can be given an orientation by a regular parameterization. If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is a regular parameterization of a C^1 curve Γ, we can define a unit vector at each point $\boldsymbol{\gamma}(t) \in \Gamma$ on the curve by taking the vector

$$\boldsymbol{T}(\boldsymbol{\gamma}(t)) = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}.$$

This is indeed tangent, since it is a scaled version of the tangent vector $\boldsymbol{\gamma}'(t)$, and it is a unit vector since

$$\|\boldsymbol{T}(\boldsymbol{\gamma}(t))\| = \frac{\|\boldsymbol{\gamma}'(t)\|}{\|\boldsymbol{\gamma}'(t)\|} = 1.$$

Two parameterizations give the same orientation if they go in the same direction.

Definition 2.11 (***Oriented curve***)**.** An *oriented curve* in $\mathbb{R}^n$ is a simple curve Γ together with an *orientation.*

A *parameterization* an oriented C^1 curve Γ is a regular path parameterizing the curve such that the velocity of the path is always pointing in the same direction of the orientation.

Given an oriented curve $\Gamma \subset \mathbb{R}^n$, one may consider its corresponding *reverse* curve $-\Gamma$ (or sometimes also denoted Γ^-), which is just the same curve but with the reverse orientation. If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is a parameterization of Γ, a parameterization of $-\Gamma$ can be found $\gamma_- : [-b,-a] \to \mathbb{R}^n$ by defining

$$\boldsymbol{\gamma}_-(t) = \boldsymbol{\gamma}(-t),$$

which is simply the *time reversal* of $\boldsymbol{\gamma}$. The velocity of the reversed path at any time t is

$$\boldsymbol{\gamma}'_-(t) = -\boldsymbol{\gamma}(-t),$$

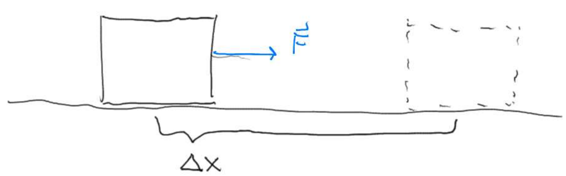
which is tangent to the curve but points in the opposite direction to the tangent found from the original path $\boldsymbol{\gamma}$.

2.1.5 Line integrals of vector fields

2.1.5.1 Work done by a force

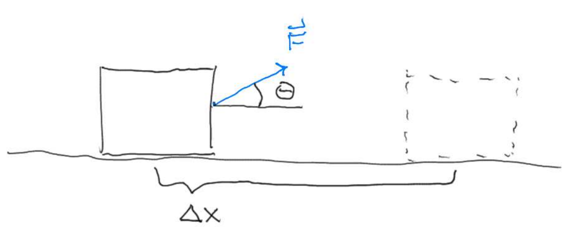
We first recall some facts about force and work from elementary physics. If a force (with magnitude $\|\boldsymbol{F}\|$) pulls an object along a surface, and the force moves the object a distance of Δx in the same direction of the the force, the amount of *work done* by the force is equal to

$$\text{work done} = \|\boldsymbol{F}\|\Delta x.$$



If the force is not necessarily pulling in the same direction that the object moves (the angle between the direction of movement and the direction of the force is θ), the work done is

$$\text{work done} = \|\boldsymbol{F}\| \sin\theta \Delta x.$$



Consider the *displacement vector* of the object to be $\Delta\boldsymbol{r} = \Delta x\,\hat{\boldsymbol{\imath}}$ and the suppose vector of the force to be $\boldsymbol{F}(\boldsymbol{r}) = F_1(\boldsymbol{r})\hat{\boldsymbol{\imath}} + F_2(\boldsymbol{r})\hat{\boldsymbol{\jmath}}$. That is

$$\Delta\boldsymbol{r} = \begin{pmatrix} \Delta x \\ 0 \end{pmatrix} \qquad \Delta\boldsymbol{F}(\boldsymbol{r}) = \begin{pmatrix} F_1(\boldsymbol{r}) \\ F_2(\boldsymbol{r}) \end{pmatrix}.$$

Then the work done can be computed as the dot product

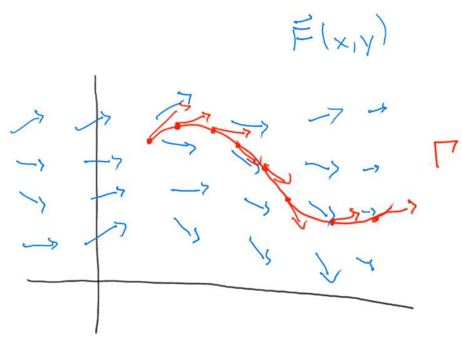
$$\text{work done} = \|\boldsymbol{F}\| \sin\theta \Delta x = \boldsymbol{F} \cdot \Delta \boldsymbol{r}.$$

This analysis assumes that the force is constant (in both direction and magnitude) and that the displacement is along a straight line. If the force changes in space and the object moves along a curve instead of a straight line, the work done is computed by a *line integral* where we add up all a bunch of infinitesimally small amounts of work done by the field on infinitely small segments of the curve. Line integrals of curves are computed by parameterizing the curve.

2.1.5.2 Definition of line integrals

Let $\boldsymbol{F} : D \to \mathbb{R}^n$ be a vector field defined on a region $D \subset \mathbb{R}^n$ and let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a C^1 path. The *line integral* of the field along the path $\boldsymbol{\gamma}$ is defined as

$$\int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt.$$



Note that line integrals of vector fields are *direction dependent*. If we reverse the path and instead take the line integral of $\boldsymbol{F}$ along the reverse path $\boldsymbol{\gamma}_-(t)$, we get

$$\begin{aligned}\int_{-b}^{-a} \boldsymbol{F}(\boldsymbol{\gamma}_-(t)) \cdot \boldsymbol{\gamma}_-'(t)\, dt &= \int_{-b}^{-a} \boldsymbol{F}(\boldsymbol{\gamma}(-t)) \cdot \boldsymbol{\gamma}'(t)\, dt \\ &= -\int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(s)) \cdot \boldsymbol{\gamma}'(s)\, ds\end{aligned}$$

where in the last line we make the change of variables $s = -t$. This is exactly the *negative* of the value of the line integral of $\boldsymbol{F}$ along the original path $\boldsymbol{\gamma}$.

On the other hand, if two regular C^1 paths $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ parameterize the same simple curve Γ with the same orientation, one has that

$$\int_{a_1}^{b_1} \boldsymbol{F}(\boldsymbol{\gamma}_1(t)) \cdot \boldsymbol{\gamma}_2'(t)\, dt = \int_{a_2}^{b_2} \boldsymbol{F}(\boldsymbol{\gamma}_2(t)) \cdot \boldsymbol{\gamma}_2'(t)\, dt.$$

Thus, given an *oriented* C^1 curve Γ, we may unambiguously define the *line integral of* $\boldsymbol{F}$ *along the curve* Γ as

$$\boxed{\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt}$$

for any orientation preserving parameterization $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ of Γ.

Example 2.12. Consider the vector field defined by $\boldsymbol{F}(x,y) = (y, -x)$.

- For the path $\boldsymbol{\gamma}_1 : [0, \pi/2] \to \mathbb{R}^2$ defined by $\boldsymbol{\gamma}_1(t) = (\cos t, \sin t)$, the line integral of $\boldsymbol{F}$ along the curve Γ_1 defined by this path can be computed as follows. The velocity of the path is

$$\boldsymbol{\gamma}_1'(t) = (-\sin t, \cos t)$$

and the field evaluated at each point on the curve is

$$\boldsymbol{F}(\boldsymbol{\gamma}_1(t)) = \boldsymbol{F}(\cos t, \sin t) = (\sin t, -\cos t).$$

The line integral is therefore

$$\begin{aligned}\int_{\Gamma_1} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{\pi/2} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\, dt \\ &= -\int_0^{\pi/2} (\sin^2 t + \cos^2 t)\, dt = -\int_0^{\pi/2} 1\, dt = -\frac{\pi}{2}.\end{aligned}$$

beginitemize

- For the path $\boldsymbol{\gamma}_2 : [0,1] \to \mathbb{R}^2$ defined by $\boldsymbol{\gamma}_2(t) = (1-t, t)$, the line integral of $\boldsymbol{F}$ along the curve Γ_2 defined by this path can be computed as follows. The velocity of the path is

$$\boldsymbol{\gamma}_2'(t) = (-1, 1)$$

and the field evaluated at each point on the curve is

$$\boldsymbol{F}(\boldsymbol{\gamma}_2(t)) = \boldsymbol{F}(1-t, t) = (t, t-1).$$

The line integral is therefore

$$\begin{aligned}\int_{\Gamma_2} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^1 (t, t-1) \cdot (-1, 1)\, dt \\ &= \int_0^1 (t - (t-1))\, dt = -\int_0^1 1\, dt = -1.\end{aligned}$$

Line integrals along different curves are generally different, even if they have the same start and end points.

Remark. A simple curve that is piecewise C^1 can also be oriented. If we can write the curve Γ as the union of C^1 curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$, the line integral can be computed by summing up the parts as

$$\int_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\Gamma_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \cdot + \int_{\Gamma_N} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Remark. When Γ is a *closed* oriented curve, the line integral along the curve is typically denoted using a different symbol. We write

$$\oint_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{r}$$

for this line integral to indicate that the curve is closed, and we often call this the "circulation of $\boldsymbol{F}$ around Γ."

Example 2.13. Compute the work done by the force field $\boldsymbol{F}(x,y) = x^2\,\hat{\boldsymbol{\imath}} + xy\,\hat{\boldsymbol{\jmath}}$ on a particle that moves clockwise half way around the circle $x^2 + y^2 = 4$ from $(2,0)$ to the point $(-2,0)$, then back to the starting point along the x-axis.

The first part of the curve can be parameterized by the path

$$\boldsymbol{\gamma}_1(t) = (\cos t, \sin t) \qquad \text{with velocity} \quad \boldsymbol{\gamma}_1'(t) = (-\sin t, \cos t)$$

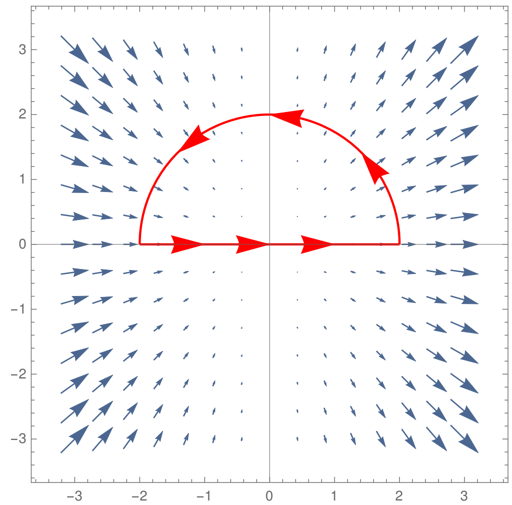


Figure 2.11: The vector field and the oriented curve in Example 2.13.

for $t \in [0, \pi]$, while the second part of the curve can be parameterized by

$$\boldsymbol{\gamma}_2(t) = (t, 0) \qquad \text{with velocity} \quad \boldsymbol{\gamma}_2'(t) = (1, 0)$$

for $t \in [-2, 2]$. The line integrals along each of the pieces can be computed as

$$\begin{aligned}\int_{\Gamma_1} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{\pi} \boldsymbol{F}(\boldsymbol{\gamma}_1(t)) \cdot \boldsymbol{\gamma}_1'(t)\, dt \\ &= \int_0^{\pi} (\cos^2 t, \cos t \sin t) \cdot (-\sin t, \cos t)\, dt = \int_0^{\pi} (\cos^2 t \sin t - \cos^2 t \sin t)\, dt = 0,\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_2} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{-2}^{2} \boldsymbol{F}(\boldsymbol{\gamma}_2(t)) \cdot \boldsymbol{\gamma}_2'(t)\, dt \\ &= \int_{-2}^{2} (t^2, 0) \cdot (1, 0)\, dt = \int_{-2}^{2} t^2\, dt = \frac{16}{3}.\end{aligned}$$

The line integral along the entire closed curve can be computed as

$$\oint_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\Gamma_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{\Gamma_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{16}{3}.$$

2.1.6 The gradient and conservative vector fields

Definition 2.14 (***Conservative field***)**.** A vector field $\boldsymbol{F} : D \to \mathbb{R}^n$ in a region $D \subset \mathbb{R}^n$ is said to be *conservative* if there is a differentiable scalar field $\Psi : D \to \mathbb{R}$ such that $\nabla\Psi(\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r})$ at all points $\boldsymbol{r} \in D$. The scalar field Ψ is called the *scalar potential* of $\boldsymbol{F}$.

The name 'conservative' comes from 'conservation' laws in physics. For example, if $\boldsymbol{F}$ is a force field that arises from some function Ψ that defines the potential energy of a particle at a point in space, certain *conservation of energy* laws can be derived.

Recall that, for a function $f : \mathbb{R}^n \to \mathbb{R}$, if all of the second order partial derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

exist and are continuous for all indices $i, j \in \{1, \ldots, n\}$ (i.e., if Ψ is C^2), then it does not matter in which order we take the derivatives with respect to the independent variables x_i and x_j. That is,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

This fact is useful for determining if a vector field is conservative.

First look at an example in $\mathbb{R}^3$. Let $\boldsymbol{F} : D \to \mathbb{R}^3$ be a conservative vector field that is the gradient of some C^2 scalar potential field Ψ. The components of the vector field are

$$\boldsymbol{F}(x, y, z) = \big(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\big)$$

whereas the components of the gradient of the scalar field are

$$\nabla\Psi(x, y, z) = \left(\frac{\partial \Psi}{\partial x}\bigg|_{(x,y,z)}, \frac{\partial \Psi}{\partial y}\bigg|_{(x,y,z)}, \frac{\partial \Psi}{\partial z}\bigg|_{(x,y,z)} \right).$$

Since $\boldsymbol{F} = \nabla\Psi$, we can equate each of the components of these vector fields:

$$F_1 = \frac{\partial \Psi}{\partial x}, \qquad F_2 = \frac{\partial \Psi}{\partial y}, \qquad \text{and} \quad F_3 = \frac{\partial \Psi}{\partial z}.$$

Since Ψ is C^2, all of its second order partial derivatives exist and are continuous, and it doesn't matter in which order we take the derivatives with respect the variables. That is,

$$\frac{\partial^2 \Psi}{\partial y \partial x} = \frac{\partial^2 \Psi}{\partial x \partial y}, \qquad \frac{\partial^2 \Psi}{\partial z \partial x} = \frac{\partial^2 \Psi}{\partial x \partial z}, \qquad \text{and} \qquad \frac{\partial^2 \Psi}{\partial y \partial z} = \frac{\partial^2 \Psi}{\partial z \partial y}.$$

Now, if we take the partial derivative of the first component of $\boldsymbol{F}$ with respect to the second variable, we get

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial x} = \frac{\partial^2 \Psi}{\partial y \partial x} = \frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial y} = \frac{\partial F_2}{\partial x}.$$

Taking the derivative of F_1 with respect to z and the derivative of F_2 with respect to z gives us similar results. All in all, if $\boldsymbol{F}$ is conservative and is the gradient of some C^2 potential field, it must hold that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{2.2}$$

In fact, if $\boldsymbol{F}$ is C^1, then $\boldsymbol{F}$ is the gradient of some C^1 scalar potential field Ψ *if and only if* the derivatives of its components satisfy the equations in (2.2). This rule holds in general for all dimensions.

Theorem 2.15. *Let* $\boldsymbol{F} : D \to \mathbb{R}^n$ *be a* C^1 *vector field on a region* $D \subset \mathbb{R}^n$. *It holds that* $\boldsymbol{F}$ *is conservative if and only if it holds that*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

for all indices $i, j \in \{1, \ldots, n\}$.

This rule can be used to check if a vector field is conservative and help us find a scalar potential if it is.

Example 2.16. Here we find scalar potentials for conservative vector fields.

1. Consider the vector field on $\mathbb{R}^2$ given by $\boldsymbol{F}(x, y) = (2xy, 1 + x^2)$. The components of this vector field are $F_1(x, y) = 2xy$ and $F_2(x, y) = 1 + x^2$ such that $\boldsymbol{F}(x, y) = (F_1(x, y), F_2(x, y))$. We check the partial derivatives of the components,

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x \qquad \text{and} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(1 + x^2) = 2x.$$

These are equal so $\boldsymbol{F}$ must be conservative and we can find a scalar potential. That is, there is a scalar field $\Psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla\Psi$. This scalar field must satisfy

$$\frac{\partial \Psi}{\partial x} = F_1(x, y) = 2xy \qquad \text{and} \qquad \frac{\partial \Psi}{\partial y} = 1 + x^2.$$

Taking the integral of the first equation with respect to x (and holding y constant) yields

$$\int \frac{\partial \Psi}{\partial x}\, dx = \int 2xy\, dx \quad \Longrightarrow \quad \Psi(x, y) = x^2 y + f(y),$$

where $f(y)$ is constant with respect to x and depends only on the value of y. To find what this function is, note that we can take the derivative of Ψ with respect to y which must yield F_2. That is,

$$1 + x^2 = F_2(x, y) = \frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}(x^2 y + f(y)) = x^2 + f'(y),$$

and thus $1 + x^2 = f'(y) + x^2$. We see that the function f must satisfy $f'(y) = 1$, so f must be of the form $f(y) = y + c$ for some constant c. Hence a scalar potential for $\boldsymbol{F}$ is

$$\Psi(x, y) = x^2 y + y + c.$$

2. A similar strategy can be used for vector fields on $\mathbb{R}^3$. Consider the vector field on $\mathbb{R}^3$ given by

$$\boldsymbol{F}(x, y, z) = y\,\hat{\boldsymbol{\imath}} + (z\cos(yz) + x)\,\hat{\boldsymbol{\jmath}} + y\cos(yz)\,\hat{\boldsymbol{k}}.$$

The components of this vector field $\boldsymbol{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ are

$$F_1(x, y, z) = y, \qquad F_2(x, y, z) = z\cos(yz) + x, \qquad \text{and} \qquad F_3(x, y, z) = z\cos(yz).$$

We first check to see that this vector field is conservative. Note that

$$\begin{aligned} &\frac{\partial F_1}{\partial y} = 1 \qquad \text{and} \qquad \frac{\partial F_2}{\partial x} = 1 \\ &\frac{\partial F_1}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial F_3}{\partial x} = 0 \\ &\frac{\partial F_2}{\partial y} = -yz\sin(yz) \qquad \text{and} \qquad \frac{\partial F_3}{\partial y} = -yz\sin(yz), \end{aligned}$$

so we can find a scalar potential Ψ. This scalar potential must satisfy

$$\frac{\partial \Psi}{\partial x} = y \qquad \frac{\partial \psi}{\partial y} = z\cos(yz) + x, \qquad \text{and} \qquad \frac{\partial \Psi}{\partial z} = y\cos(yz).$$

Taking the antiderivative of the first equation with respect to x (while holding y and z constant) yields

$$\int \frac{\partial \Psi}{\partial x}\, dx = \int y\, dx \qquad \Longrightarrow \qquad \Psi(x, y, z) = xy + f(y, z),$$

where $f(y,z)$ is constant with respect to x and depends only on the values of y and z. Taking the derivative of this with respect to y yields

$$F_2(x,y,z) = \frac{\partial \Psi}{\partial y}\Big|_{(x,y,z)} = \frac{\partial}{\partial y}\left(xy + f(y,z)\right) = x + \frac{\partial f}{\partial y}\Big|_{(y,z)}.$$

Since this must be equal to $F_2(x,y,z) = z\cos(yz) + x$, we find that f must satisfy

$$\frac{\partial f}{\partial y}\Big|_{(y,z)} = z\cos(yz).$$

Taking the antiderivative of this equation with respect to y (and holding z constant) yields

$$\int \frac{\partial f}{\partial y}\,dy = \int z\cos(yz)\,dy \qquad \Longrightarrow \qquad f(y,z) = \sin(yz) + c$$

for some constant c. Hence the scalar potential has the form

$$\Psi(x,y,z) = xy + \sin(yz) + c.$$

Finally, we check that

$$\frac{\partial \Psi}{\partial z} = \frac{\partial}{\partial z}(xy + \sin(yz) + c) = y\cos(yz) = F_3(x,y,z),$$

so Ψ is indeed a scalar potential for $\boldsymbol{F}$.

2.1.7 A chain rule for scalar fields

Suppose $f : D \to \mathbb{R}$ is a C^1 scalar field on a region $D \subset \mathbb{R}^n$ and suppose we have a differentiable path $\boldsymbol{\gamma}$ in D. The value of $f(\boldsymbol{\gamma}(t))$ indicates the value of f at the point on the path $\boldsymbol{\gamma}(t)$ at time t. What is the rate of change of $f(\boldsymbol{\gamma}(t))$ with respect to time?

In $\mathbb{R}^3$, we can make sense of this by defining the path in terms of its coordinate functions

$$\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$$

whose velocity is given by

$$\boldsymbol{\gamma}'(t) = (x'(t), y'(t), z'(t)).$$

At a point $\boldsymbol{r} = (x,y,z)$, the value of the scalar field is $f(\boldsymbol{r}) = f(x,y,z)$. Using the chain rule for multiple variables, if the variables x, y, and z all depend on some underlying variable t, we have

$$\frac{d}{dt} f\big(x,y,z\big) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Hence

$$\begin{aligned}
\frac{d}{dt} f\big(\boldsymbol{\gamma}(t)\big) = \frac{d}{dt} f\big(x(t), y(t), z(t)\big) &= \frac{\partial f}{\partial x}\Big|_{\boldsymbol{\gamma}(t)} \frac{d}{dt}x(t) + \frac{\partial f}{\partial y}\Big|_{\boldsymbol{\gamma}(t)} \frac{d}{dt}y(t) + \frac{\partial f}{\partial z}\Big|_{\boldsymbol{\gamma}(t)} \frac{d}{dt}z(t) \\
&= \left(\frac{\partial f}{\partial x}\Big|_{\boldsymbol{\gamma}(t)}, \frac{\partial f}{\partial y}\Big|_{\boldsymbol{\gamma}(t)}, \frac{\partial f}{\partial z}\Big|_{\boldsymbol{\gamma}(t)} \right) \cdot (x'(t), y'(t), z'(t)) \\
&= \nabla f(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t).
\end{aligned}$$

We can write the derivative of f along $\boldsymbol{\gamma}$ as the dot product of the gradient and the velocity! This holds in general for higher dimensions as well.

Theorem 2.17. *Let $f : D \to \mathbb{R}$ be a C^1 scalar field on a region $D \subset \mathbb{R}^n$ and let $\boldsymbol{\gamma}$ be a C^1 path in D. For all t it holds that*

$$\frac{d}{dt} f(\boldsymbol{\gamma}(t)) = \nabla f(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t). \tag{2.3}$$

2.1.8 Fundamental theorem of line integrals

First recall the Fundamental Theorem of Calculus. For a differentiable function $f : [a, b] \to \mathbb{R}$, the integral of its derivative is

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

Now let $f : D \to \mathbb{R}$ be a C^1 scalar field on a region $D \subset \mathbb{R}^n$ and let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a C^1 path in D. By the fundamental theorem of calculus, we have

$$\begin{aligned} \int_a^b \nabla f(\boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'(t)\, dt &= \int_a^b \frac{d}{dt} f(\boldsymbol{\gamma}(t))\, dt \\ &= f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)). \end{aligned}$$

Remark. Suppose $\boldsymbol{F} : D \to \mathbb{R}^n$ is a conservative field on a domain $D \subset \mathbb{R}^n$ and suppose P_0 and P_1 are two points in D. Since $\boldsymbol{F}$ is conservative, there is a scalar potential function such that $\boldsymbol{F} = \nabla\Psi$. If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ is any path from P_0 to P_1 (i.e., $\boldsymbol{\gamma}(a) = P_0$ and $\boldsymbol{\gamma}(a) = P_1$), then

$$\int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \Psi(P_1) - \Psi(P_0).$$

Since it does not patter which path we use to connect P_0 to P_1, it is common to write the resulting integral as

$$\int_{P_0}^{P_1} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

We say that line integrals of conservative vector fields are *independent of path*, since they only depend on the start and end points of an oriented curve and not the specific path that connects those points.

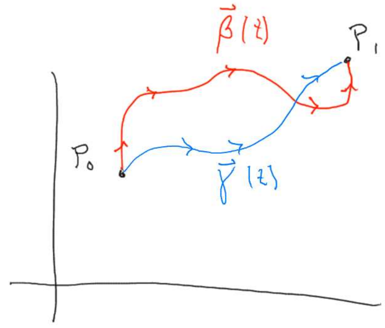


Figure 2.12: If two paths $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ have the same start and end points, the line integral of a conservative vector field will be the same regardless of which path we take.

If Γ is a *closed* curve, then the start and end points are the same. Thus,

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

for any conservative field $\boldsymbol{F}$.

Example 2.18. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\boldsymbol{F}(x, y, z) = y\,\hat{\boldsymbol{\imath}} + (z\cos(yz) + x)\,\hat{\boldsymbol{\jmath}} + y\cos(yz)\,\hat{\boldsymbol{k}}$$

and let Γ be any oriented curve that starts at $P_0 = (0, 0, 1)$ and ends at $P_1 = (1, 1, \pi)$. We can evaluate $\int_{P_0}^{P_1} \boldsymbol{F} \cdot d\boldsymbol{r}$ as follows. From Example 2.16(2), we see that $\boldsymbol{F}$ is conservative with scalar potential given by $\Psi(x, y, z) = xy + \sin(yz)$. Therefore,

$$\int_{P_0}^{P_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \Psi(P_1) - \Psi(P_0) = \Psi(1, 1, \pi) - \Psi(0, 0, 1) = (1 + \sin\pi) - (0 + \sin 0) = 1.$$

Theorem 2.19. *Let $f : D \to \mathbb{R}$ be a C^1 vector field on $D \subset \mathbb{R}^n$.*

1. *For all closed oriented C^1 curves Γ in D, it holds that*

$$\oint_\Gamma \nabla f \cdot \boldsymbol{r} = 0$$

2. *If Γ is an oriented curve in D from a point P_0 to a point P_1, it holds that*

$$\int_\Gamma \nabla f \cdot d\boldsymbol{r} = f(P_1) - f(P_0)$$