ECE 206 – University of Waterloo — Fall 2019

Lecture notes for Week 3

*Lecturer: Mark Girard* — *September 17, 2019*

## 3.1 Two-dimensional integration

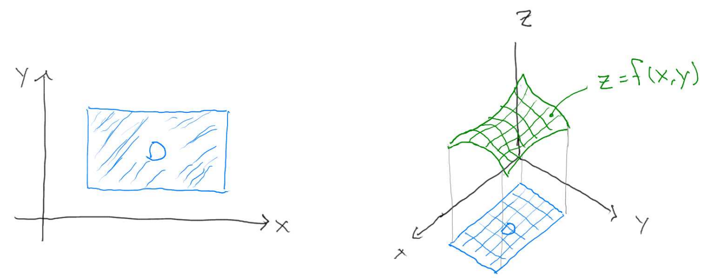



Figure 3.1: The *graph* of a function $f : D \to \mathbb{R}$ on a region $D \subseteq \mathbb{R}^2$ is the surface resting in space above the plane.

Consider a region $D \subset \mathbb{R}^2$ in the plane and a scalar field $f : D \to \mathbb{R}$. The surface that is the *graph* of of $f$ can be viewed as sitting above the region $D$ in the $xy$-plane. We can now consider the resulting 3-dimensional *solid* that is bounded below by $D$ and bounded above by the surface of the graph of $f$. A *two-dimensional integral* can be interpreted as the volume of this resulting solid.

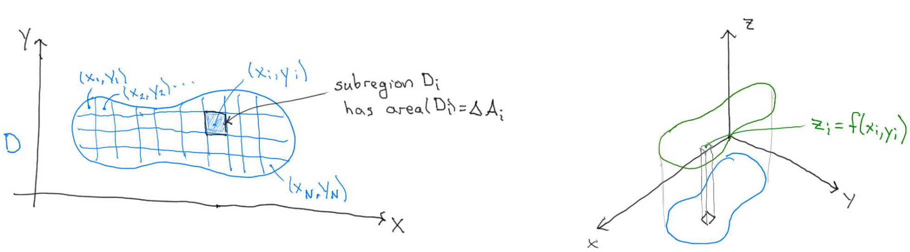



Figure 3.2: A partitioning of a region $D$ into $N$ smaller subregions can be used to approximate the volume of the solid between the $xy$-plane and the surface of the graph of $f : D \to \mathbb{R}$.

We can approximate the volume of this solid as follows. Partition the region $D$ in to $N$ different subregions $D_1, \ldots, D_N$, where $i^{\text{th}}$ subregion $D_i$ has area $\Delta A_i$, and pick a point $(x_i, y_i)$ in each subregion. The volume of the part of the solid that sits over $D_i$ can approximated as the volume of a column whose height is $f(x_i, y_i)$ and whose base is $D_i$ with area $\Delta A_i$. The volume of each piece is therefore approximated by

$$\text{volume of solid over } D_i \approx f(x_i, y_i)\,\Delta A_i,$$

and the volume of the entire solid is approximated by summing up all of these small pieces of volume

$$\text{volume of entire solid } \approx \sum_{i=1}^{N} f(x_i, y_i)\,\Delta A_i. \tag{3.1}$$

Partitioning the region $D$ into smaller and smaller pieces yields a better and better approximation for the true value of this volume. If this approximation approaches a limiting value as $N \to \infty$ and $\Delta A_i \to 0$ (i.e., partitioning the region into more and more pieces, each with area approaching zero as the number of pieces in the partition tends to infinity), the value of this limit is expressed as a *two-dimensional integral*:

$$\iint_D f\, dA = \lim_{\substack{N\to\infty \\ \Delta A_i \to 0}} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i.$$

The double integration sign reminds us that this integral is really over two variables and the subscript $D$ tells us over which region in the plane we are taking the integral.

Actually computing the value of this integral is not done by taking this limit, but rather by using standard techniques of integration. However, the integral can only be computed straightforwardly if the region of integration is *simple*.

**Remark.** The *area* of a region $D \subset \mathbb{R}^2$ can be expressed as a two-dimensional integral by taking the constant function $f(x, y) = 1$ and evaluating the resulting integral

$$\boxed{\text{area}(D) = \iint_D dA}$$

over the region.

### 3.1.1 Simple regions

**Definition 3.1** (Simple and regular regions)**.** Let $D \subset \mathbb{R}^2$ be a region in the plane.

- The region is said to be $x$-*simple* if it can be defined all points $(x, y)$ having $x$-values bounded between two constants $x_0$ and $x_1$ and $y$-values bounded between $g_0(x)$ and $g_1(x)$, where $g_0$ and $g_1$ are some continuous functions on the interval $[x_0, x_1]$

$$D = \{(x, y) \in \mathbb{R}^2 \mid x_0 \le x \le x_1,\ g_0(x) \le y \le g_1(x)\}. \tag{3.2}$$

- The region is said to be $y$-*simple* if it can be defined all points $(x, y)$ having $y$-values bounded between two constants $y_0$ and $y_1$ and $x$-values bounded between $h_0(x)$ and $h_1(x)$, where $h_0$ and $h_1$ are some continuous functions on the interval $[y_0, y_1]$

$$D = \{(x, y) \in \mathbb{R}^2 \mid y_0 \le y \le y_1,\ h_0(y) \le x \le h_1(y)\}. \tag{3.3}$$

It is *simple* if it is either $x$- or $y$-simple, and it is *regular* if it is both $x$- and $y$-simple

Two-dimensional integrals over simple regions can be computed by integrating over each of the variables separately. For an $x$-simple region, we may first integrate with respect to $y$ while holding $x$ constant, using the limits of integration to be $g_0(x)$ and $g_1(x)$, then integrate the resulting expression with respect to $x$

$$\iint_D f\, dA = \int_{x_0}^{x_1} \left( \int_{g_0(x)}^{g_1(x)} f(x, y)\, dy \right) dx.$$

Integration over a $y$-simple region is analogous, where we first integrate with respect to $x$ while holding $y$ constant, using the limits of integration to be $h_0(x)$ and $h_1(x)$, then integrate the resulting expression with

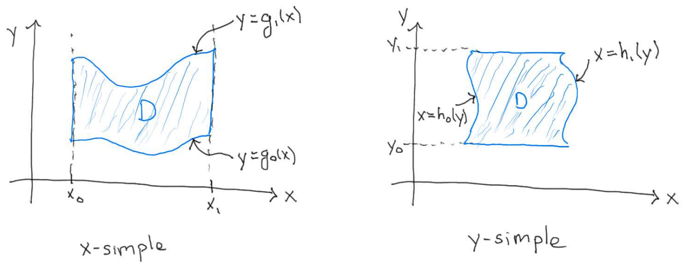



Figure 3.3: Examples of $x$-simple and $y$-simple regions.

respect to $y$

$$\iint_D f\,dA = \int_{y_0}^{y_1}\left(\int_{h_0(y)}^{h_1(y)} f(x,y)\,dx\right)\,dy.$$

In either case, we always perform the 'inside' integration first, which eliminates the 'inside' variable from the resulting expression. The resulting expression after evaluating the double integral is a number and should not have any $x$'s or $y$'s in it.

**Example 3.2.** Let $D$ be the region in the plane whose that is bounded below $y = 0$ and bounded above by $y = \sqrt{1+\cos x}$ for $0 \le x \le 2\pi$, and consider the function defined by $f(x, y) = 2y$. We can integrate $f$ over the region $D$, shown in the figure below, by viewing $D$ as an $x$-simple region and performing a double integration.

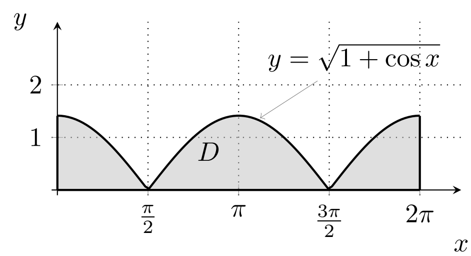



Figure 3.4: The region of integration in Example 3.2

The value of the resulting two-dimensional integral computed using is double integral is

$$\begin{aligned}
\iint_D f\,dA &= \int_0^{2\pi}\left(\int\int_0^{\sqrt{1+\cos x}} 2y\,dy\right)\,dx \\
&= \int_0^{2\pi}\left(y^2\Big|_0^{\sqrt{1+\cos x}}\right)\,dx \\
&= \int_0^{2\pi}(1+\cos x)\,dx \\
&= 2\pi,
\end{aligned}$$

where we note that $\displaystyle\int_0^{2\pi} \cos x\,dx = 0$.

A region is *regular* if it is both $x$- and $y$-simple. One example of a regular region is the unit disc

$$D = \{(x, y) \mid x^2 + y^2 = 1\}.$$

As an $x$-simple region, the unit disc is the set of all points $(x, y)$ with $x$-value in the range $-1 \leq x \leq 1$ and (for a fixed value of $x$) having $y$-value in the range $-\sqrt{1-x^2} \leq y \leq +\sqrt{1-x^2}$. Meanwhile, as a $y$-simple region, the unit disc is the set of all points $(x, y)$ with $y$-value in the range $-1 \leq y \leq 1$ and (for a fixed value of $y$) having $x$-value in the range $-\sqrt{1-y^2} \leq x \leq +\sqrt{1-y^2}$. The functions that define the boundaries as an $x$-simple region ($g_0$ and $g_1$) will generally be different from the functions that define the boundaries as a $y$-simple region ($h_0$ and $h_1$).

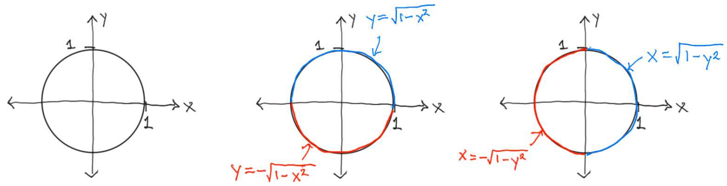



Figure 3.5: The unit disc is an example if a regular region.

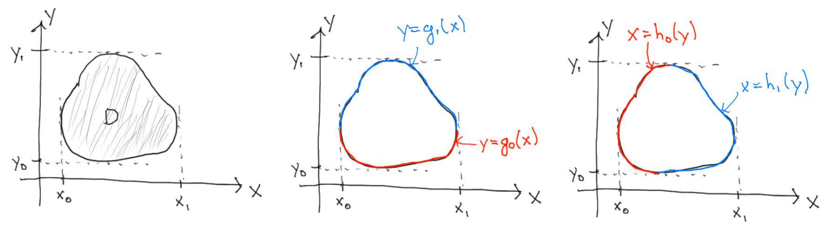



Figure 3.6: A regular region can be described as both an $x$- and $y$-simple region.

### 3.1.2 Fubini's Theorem

If a region is regular, it does not matter which order we perform the integration in! This fact is known as *Fubini's theorem.*

**Theorem 3.3** (Fubini's Theorem)**.** *Let $D \subseteq \mathbb{R}^2$ be a regular region in the plane of the form in* (3.2) *and* (3.3)*, and let $f : D \to \mathbb{R}$ be a function. If the two-dimensional integral of $f$ over $D$ exists, it can be computed by either integrating with respect to $y$ first or with respect to $x$ first:*

$$\iint_D f\,dA = \int_{x_0}^{x_1} \left( \int_{g_0(x)}^{g_1(x)} f(x,y)\,dy \right) dx = \int_{y_0}^{y_1} \left( \int_{h_0(y)}^{h_1(y)} f(x,y)\,dx \right) dy.$$

Sometimes a two-dimensional integral over a regular region can't be computed exactly when integrating the variables one way, but can be integrated when we switch the order of integration. The following example illustrates this idea.

**Example 3.4.** We can use Fubini's theorem to switch the order of integration when evaluating the integral

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx.$$

This is an example of a two-dimensional integral of the function $f(x, y) = e^{y^3}$ over some region $D$ in the plane. A straightforward attempt at computing this integral runs into trouble when trying to integrate first with respect to $y$, as we cannot find the antiderivative of $e^{y^3}$! At this point we might give up and attempt to get an approximation of this integral by integrating it numerically with a computer. However, the region of integration is both $x$- and $y$-simple, and by switching the order of integration we can express this integral in a different way. In fact, in this case when integrating over $x$ first instead of $y$, we obtain a two-dimensional that *can* be computed straightforwardly. We first need to actually describe the region of integration in this problem. As an $x$-simple region, the bounds on the $x$-variable are $0 \leq x \leq 1$, while the lower and upper bounds of the region $D$ are $y = \sqrt{x}$ and $y = 1$. This region is depicted in Figure 3.7.

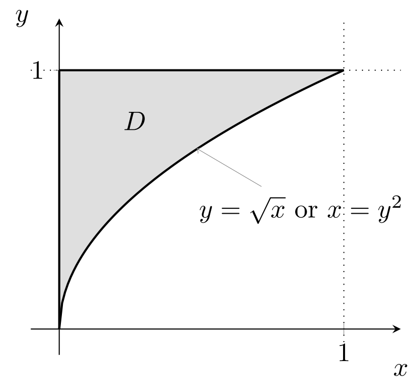



Figure 3.7: The region of integration in Example 3.4

We may alternatively describe this region as $y$-simple with lower and upper boundaries given by $0 \leq y \leq 1$ with left and right boundaries given by $x = 0$ and $x = y^2$. The integral can now be expressed as

$$\begin{aligned}\iint_D f\, dA &= \int_0^1 \left( \int_0^{y^2} e^{y^3}\, dx \right) dy \\ &= \int_0^1 \left( x e^{y^3} \Big|_{x=0}^{x=y^2} \right) dy = \int_0^1 y^2 e^{y^3}\, dy = \frac{1}{3} e^{y^3} \Big|_0^1 = \frac{e-1}{3},\end{aligned}$$

which can actually be computed.

## 3.2 Connected, simply connected, and boundaries

**Definition 3.5.** A region $D \subseteq \mathbb{R}^2$ is *connected* if, for every pair of points $P_0, P_1 \in D$, there is a continuous path in $D$ with start point $P_0$ and end point $P_1$.

It is also important to discriminate between regions that have 'holes' or not. Intuitively speaking, a region $D$ is *simply connected* if it does not have any 'holes'. This can only happen if any closed loop in $D$ can be shrunk to a single point without ever leaving the region $D$.

**Definition 3.6.** A connected region $D \subseteq \mathbb{R}^2$ is *simply connected* if every closed curve in $D$ can be continuously shrunk to a point.

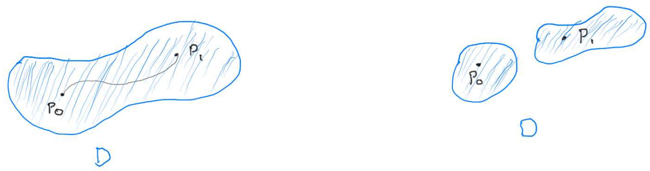



Figure 3.8: Depictions of a connected region and a disconnected region in $\mathbb{R}^2$.

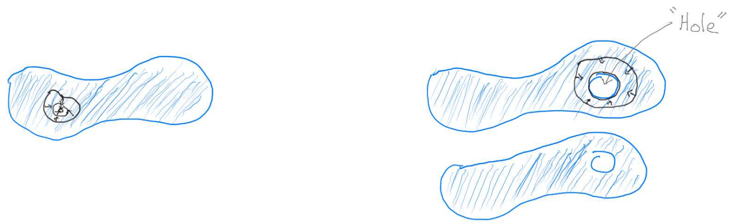



Figure 3.9: Depictions of a simply connected region and a non-simple connected region in $\mathbb{R}^2$.

Given a simply connected region $D \subset \mathbb{R}^2$ in the plane, we can talk about its *boundary*, which is a simple closed curve. The boundary of $D$ is the curve denoted by

$$\text{boundary of } D = \partial D.$$

Meanwhile, any simple closed curve $\Gamma$ always encloses a region of the plane whose boundary is $\partial D = \Gamma$. The boundary of a region can be given an orientation. If the region $D$ is a simply connected region that is enclosed by some simple closed curve, the orientation of the boundary is always taken to be *counter clockwise* by default.

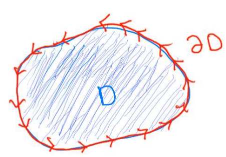



Figure 3.10: The boundary $\partial D$ of a simply connected region $D$ is an oriented curve with orientation always taken to be counter clockwise.

## 3.3 Green's Theorem

We are now going to learn an extremely powerful and useful theorem of multivariable calculus, called *Green's Theorem in the plane*. This is essentially a two-dimensional result so our focus will be exclusively on fields in $\mathbb{R}^2$, but, as we will later see, this two-dimensional result is an essential tool for establishing the main results on three dimensional vector calculus (such as Stokes' theorem and Gauss' theorem) which are indispensable for physics and engineering.

We'll start by stating Green's Theorem in its full generality before proving it for regular regions. In short, Green's theorem states that there is a relation between the *line integral* of a vector field around the boundary of the region to a *two-dimensional integral* over the interior of that same region.

**Theorem 3.7** (Green's Theorem)**.** *Let $D \subseteq \mathbb{R}^2$ be a connected region in the plane and let $\boldsymbol{F} : D \to \mathbb{R}^2$ be a vector field that is $C^1$ on all of $D$ with components $\boldsymbol{F}(x, y) = (F_1(x, y), F_1(x, y))$. It holds that*

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

We can derive a proof of Green's Theorem by restricting our attention to regular regions then show how we can extend it to arbitrary regions.

### 3.3.1 Derivation of Green's Theorem for regular regions

Suppose that $D \subset \mathbb{R}^2$ is a regular region and let $\boldsymbol{F} : D \to \mathbb{R}^2$ be a $C^1$ vector field. We'll first the vector field into its components

$$\begin{aligned} \boldsymbol{F}(x, y) = (F_1(x, y), F_2(x, y)) &= (F_1(x, y), 0) + (0, F_2(x, y)) \\ &= \boldsymbol{F}_1(x, y) + \boldsymbol{F}_2(x, y) \end{aligned}$$

where $\boldsymbol{F}_1(x, y) = (F_1(x, y), 0)$ and $\boldsymbol{F}_2(x, y) = (0, F_2(x, y))$ are the vector fields that have only one nonzero component. As $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$, the line integral around the boundary of $D$ can now be split into

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\partial D} \boldsymbol{F_1} \cdot d\boldsymbol{r} + \oint_{\partial D} \boldsymbol{F_2} \cdot d\boldsymbol{r}.$$

We can therefore prove Green's theorem if we can show that

$$\oint_{\partial D} \boldsymbol{F_2} \cdot d\boldsymbol{r} = \iint_D \frac{\partial F_2}{\partial x} \, dA \qquad \text{and} \qquad \oint_{\partial D} \boldsymbol{F_1} \cdot d\boldsymbol{r} = - \iint_D \frac{\partial F_1}{\partial x} \, dA. \tag{3.4}$$

We will also need to decompose the boundary of $D$ in two different ways. Since we have assumed the region $D$ is regular, it can be viewed as both an $x$- and $y$-simple region.

- As an $x$-simple region, the lower and upper boundaries of $D$ can be viewed as the graphs of some continuous functions $g_0$ and $g_1$ from $x_0$ to $x_1$. These parts of $\partial D$ can be parameterized by the paths

$$\boldsymbol{\beta}_0(t) = (t, g_0(t)) \qquad \text{and} \qquad \boldsymbol{\beta}_1(t) = (t, g_1(t)) \qquad \text{for } x_0 \leq t \leq x_1, \tag{3.5}$$

  and we can define $\mathrm{B}_0$ and $\mathrm{B}_1$ to be the resulting oriented curves traced out by these paths.

- As a $y$-simple region, the left and right boundaries of $D$ can be viewed as the graphs of some other continuous functions $h_0$ and $h_1$ from $y_0$ to $y_1$. These parts of $\partial D$ can be parameterized by the paths

$$\boldsymbol{\gamma}_0(t) = (h_0(t), t) \qquad \text{and} \qquad \boldsymbol{\gamma}_1(t) = (h_1(t)) \qquad \text{for } y_0 \leq t \leq y_1, \tag{3.6}$$

  and we can define $\Gamma_0$ and $\Gamma_1$ to be the resulting oriented curves traced out by these paths.

We can now decompose the boundary of $D$ in two different ways as

$$\partial D = (-\Gamma_0) \cup \Gamma_1 = \mathrm{B}_0 \cup (-\mathrm{B}_1),$$

where we must take the *reverse* of $\Gamma_0$ and $\mathrm{B}_1$ to make sure that the orientations align correctly with the orientation of $\partial D$. (See Figure 3.11.)

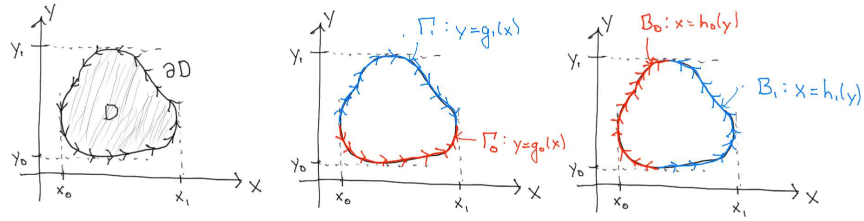



Figure 3.11: Two different decompositions of the boundary $\partial D$ of a regular region $D$.

We are now ready to prove the two equations in (3.4) to prove Green's Theorem for regular domains. To prove the first equation in (3.4), we split up the line integral around the oriented boundary curve $\partial D$ into two integrals, one along $\Gamma_1$ and another along $\Gamma_0$ with its orientation reversed

$$\oint_{\partial D} \boldsymbol{F}_2 \cdot d\boldsymbol{r} = \int_{\Gamma_1} \boldsymbol{F}_2 \cdot d\boldsymbol{r} + \int_{-\Gamma_0} \boldsymbol{F}_2 \cdot d\boldsymbol{r}.$$

Using the parameterizations for $\Gamma_0$ and $\Gamma_1$ from (3.6), we have

$$\begin{aligned}
\oint_{\partial D} \boldsymbol{F}_2 \cdot d\boldsymbol{r} &= \int_{\Gamma_1} \boldsymbol{F}_2 \cdot d\boldsymbol{r} - \int_{\Gamma_0} \boldsymbol{F}_2 \cdot d\boldsymbol{r} \\
&= \int_{y_0}^{y_1} \boldsymbol{F}_2(\boldsymbol{\gamma}_1(t)) \cdot \boldsymbol{\gamma}_1'(t)\, dt - \int_{y_0}^{y_1} \boldsymbol{F}_2(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_0'(t)\, dt \\
&= \int_{y_0}^{y_1} (0, F_2(h_1(t), t)) \cdot (h_1'(t), 1)\, dt - \int_{y_0}^{y_1} (0, F_2(h_0(t), t)) \cdot (h_0'(t), 1)\, dt \\
&= \int_{y_0}^{y_1} \big(F_2(h_1(t), t) - F_2(h_0(t), t)\big)\, dt \\
&= \int_{y_0}^{y_1} \big(F_2(h_1(y), y) - F_2(h_0(y), y)\big)\, dy \\
&= \int_{y_0}^{y_1} \left( F_2(x, y)\Big|_{x=h_0(y)}^{x=h_1(y)} \right) dy, \qquad (3.7)
\end{aligned}$$

where in the second to last line we simply change the name of the variable of integration from $t$ to $y$. Now, recall that the Fundamental Theorem of Calculus tells us that

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a) = f(x)\Big|_a^b \qquad (3.8)$$

for any differentiable function $f$. This still works if $f$ is a multivariate function and the differentiation and integration of $f$ in (3.8) are respect to $x$. So, in (3.7) we can simplify

$$F_2(x, y)\Big|_{x=h_0(y)}^{x=h_1(y)} = \int_{h_0(y)}^{h_1(y)} \frac{\partial F_2}{\partial x}\, dx,$$

and this yields the desired result of

$$\oint_{\partial D} \boldsymbol{F_2} \cdot d\boldsymbol{r} = \int_{y_0}^{y_1} \int_{h_0(y)}^{h_1(y)} \frac{\partial F_2}{\partial x}\, dx\, dy = \iint_D \frac{\partial F_2}{\partial x}\, dA.$$

On the other hand, taking the line integral of $\boldsymbol{F}_1$ along the boundary of $D$ by splitting the curve into $\mathrm{B}_0$

and $-\mathrm{B}_1$, where we use the parameterizations in (3.5), we have

$$\begin{aligned}\oint_{\partial D} \boldsymbol{F}_1 \cdot d\boldsymbol{r} &= \int_{\mathrm{B}_0} \boldsymbol{F}_1 \cdot d\boldsymbol{r} - \int_{\mathrm{B}_1} \boldsymbol{F}_1 \cdot d\boldsymbol{r} \\ &= \int_{x_0}^{x_1} \boldsymbol{F}_1(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_0'(t)\,dt - \int_{x_0}^{x_1} \boldsymbol{F}_1(\boldsymbol{\gamma}_1(t)) \cdot \boldsymbol{\gamma}_1'(t)\,dt \\ &= \int_{x_0}^{x_1} (F_1(t, g_0), 0) \cdot (1, g_0'(t))\,dt - \int_{x_0}^{x_1} (F_1(t, g_1(t)), 0) \cdot (1, g_1'(t))\,dt \\ &= \int_{x_0}^{x_1} \big(F_1(t, g_0(t)) - F_1(t, g_1(t))\big)\,dt \\ &= -\int_{x_0}^{x_1} \big(F_1(y, g_1(y)) - F_1(y, g_0(y))\big)\,dx \\ &= -\int_{x_0}^{x_1} \left( F_1(x,y)\Big|_{y=g_0(x)}^{y=g_1(x)} \right) dx, \end{aligned} \tag{3.9}$$

where in the second to last line we again change the name of the variable, but this time from $t$ to $x$, and swap the order of the terms in the integral and add a minus sign out front. As earlier, we can use the Fundamental Theorem of Calculus to simplify

$$F_1(x,y)\Big|_{y=g_0(x)}^{y=g_1(x)} = \int_{g_0(x)}^{g_1(x)} \frac{\partial F_1}{\partial y}\,dy,$$

and the expression in (3.9) reduces to

$$\oint_{\partial D} \boldsymbol{F_1} \cdot d\boldsymbol{r} = -\int_{x_0}^{x_1} \int_{g_0(x)}^{g_1(x)} \frac{\partial F_1}{\partial y}\,dy\,dx = -\iint_D \frac{\partial F_1}{\partial y}\,dA.$$

This completes the proof.

### 3.3.2 Green's theorem for non-regular domains

The proof above only works when the region $D$ is regular, but Green's Theorem is still valid for all kinds of regions in the plane, as long as the boundary of $D$ is a piecewise $C^1$ curve and the vector field $\boldsymbol{F}$ is $C^1$ on all of $D$.

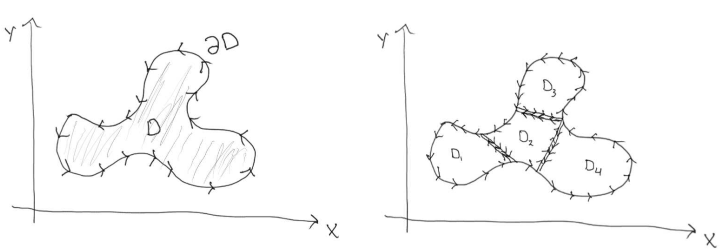



Figure 3.12: Any simply connected region can be partitioned into many smaller regular regions. We can apply Green's Theorem to each region separately and add up the results to obtain Green's Theorem for all of $D$.

If the region $D$ is simply connected, then we can partition $D$ into smaller regions, $D_1, D_2, \ldots, D_N$, each of which is a regular region (see Figure 3.12), such that

$$D = D_1 \cup D_2 \cup \cdots \cup D_N.$$

The two-dimensional integral over $D$ in Green's Theorem can then be computed by summing up the two-dimensional integrals over each of the pieces $D_i$,

$$\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \sum_{i=1}^{N} \iint_{D_i} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Meanwhile, the line integral of Green's Theorem around the boundary of $D$ can also be computed by summing up the individual line integrals around each of the boundaries of the pieces $D_i$,

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \sum_{i=1}^{N} \oint_{\partial D_i} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Indeed, piecing together the outer parts of the boundaries of each of the subregions yields the whole boundary of the entire region $D$. But the parts of the boundaries of each of the subregions $D_i$ that do *not* lie on the outer boundary $\partial D$ of the entire region are each integrated over *twice*, once in each direction. These parts of the line integrals cancel each other out, since integrating along a curve in its opposite orientation yields the *negative* of the integral along its forward orientation,

$$\int_{-\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Adding up all of the line integrals of $\boldsymbol{F}$ around the boundaries of the parts $D_i$ therefore yields the line integral of $\boldsymbol{F}$ around the whole boundary of $D$, where only the integral around $\partial D$ does not cancel out. The desired result for the entire region $D$ now follows, since Green's Theorem can be applied to each of the smaller regular regions individually,

$$\oint_{\partial D_i} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{D_i} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

and summing over $i$ yields the result.

Green's Theorem also holds for arbitrary connected regions, even if the region $D$ is not *simply* connected. However, if the region $D$ has any 'holes' then we must carefully consider what happens along the *inner* boundaries of $D$. The boundary of a region with holes is now defined as the union of a bunch of *disconnected* curves, but we must take the orientation of the inner parts of the boundaries to be *clockwise*, while the outer boundary is still taken to be counter clockwise. For example, in Figure 3.13, the boundary of the region $D$ is composed of the separate closed simple oriented curves $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$, such that

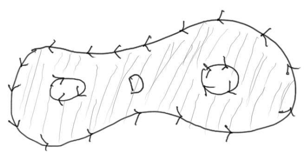
$$\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where the outer boundary $\Gamma_1$ has counter clockwise orientation and the inner boundaries $\Gamma_2$ and $\Gamma_3$ have clockwise orientation. The line integral around the entire boundary $\partial D$ can then be interpreted as the sum of the line integrals around each the separate parts,

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\Gamma_0} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{\Gamma_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{\Gamma_2} \boldsymbol{F} \cdot d\boldsymbol{r},$$

where we must make sure to take the correct orientation for each piece.

With this definition of the boundary for non-simply connected regions, we can partition any such region $D$ into many simply connected domains $D_1, \ldots, D_N$, and, as before, apply Green's Theorem to each of these subregions and sum up the results to obtain the result on all of $D$, and we interpret the line integral around the boundary of $D$ as the sum of the line integral around each of the separate parts of the boundary.



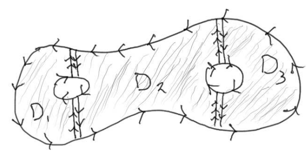




Figure 3.13: The boundary of a non-simply connected domain is the union of disjoint closed simple oriented curves, where the outer boundary has counterclockwise orientation and the inner boundaries have clockwise orientation.

Separating non-simply connected domain

Figure 3.14: We can separate non-simply connected regions into many simply connected domains and apply Green's theorem to each piece.

### 3.3.3 Applications of Green's Theorem

**Example 3.8.** Suppose $\Gamma$ is the unit circle in the plane oriented counter clockwise and let $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined in terms of its components $\boldsymbol{F}(x, y) = (F_1(x, y), F_2(x, y))$ by

$$F_1(x, y) = y + \ln(1 + x^2) \qquad \text{and} \qquad F_2(x, y) = 8x + \sin y.$$

If we were asked to compute the value of the line integral of $\boldsymbol{F}$ around the unit circle by parameterizing the curve, the resulting integral would be quite complicated! In this example it will be much simpler to evaluate this integral if we use Green's Theorem to change the line integral around a curve into a two-dimensional integral on the region enclosed by the curve. To use Green's Theorem, we must interpret the closed oriented curve $\Gamma$ as the boundary of some region. Let $D$ be the region that is the *unit disc* in the plane

$$D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$$

such that the unit circle oriented counter clockwise is exactly the boundary of $D$. Then the desired line integral can be computed using Green's Theorem as

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right) dA.$$

In this example, it turns out that the mixed partial derivatives of the comments of $\boldsymbol{F}$ are quite simple, since

$$\frac{\partial F_2}{\partial x} = 8 \qquad \text{and} \qquad \frac{\partial F_1}{\partial y} = 1$$

such that the desired integral simplifies to

$$\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right) dA = \iint_D 7 \, dA = 7 \iint_D dA = 7 \,\text{area}(D) = 7\pi,$$

where we use the fact that the area of the unit disc (the interior of a circle with radius 1) is $\text{area}(D) = \pi$.

The example above shows that a line integral of some vector field around some closed curve can be related to the *area* of the region enclosed by that curve. With a judicious choice of a $C^1$ vector field $\boldsymbol{F}(x, y)$ whose partial derivatives satisfy

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1, \tag{3.10}$$

we can use Green's Theorem to compute the area of a region $D$ by evaluating the line integral of $\boldsymbol{F}$ around $\partial D$. For this type of application of Green's Theorem, it is usually best to choose a vector field whose comments are simple so that the resulting line integral is easy to compute. A typical choice for this is the field

$$\boldsymbol{F}(x, y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$$

which clearly satisfies (3.10). This gives us a useful rule for computing the area of a region:

$$\boxed{\text{area}(D) = \frac{1}{2}\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} \qquad \text{where } \boldsymbol{F}(x, y) = (-y, x).} \tag{3.11}$$

**Example 3.9.** Consider the curve that is a *hypocycloid* in the plane defined by the equation

$$x^{2/3} + y^{2/3} = 1. \tag{3.12}$$

We can use (3.11) to compute the area of the region $D$ that is inside this curve.

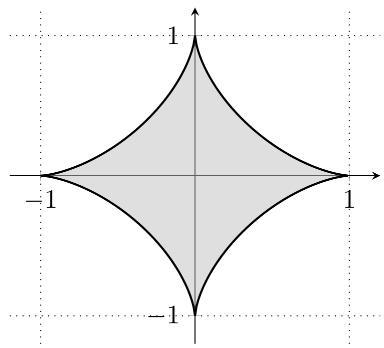



Figure 3.15: The region inside the curve that is the hypocycloid defined by (3.12). We can use Green's Theorem to compute the area contained inside this curve by evaluating a line integral.

To compute the resulting line integral, we must first provide a parameterization for this curve. Rewriting the equation defining this hypocycloid as

$$\left(x^{1/3}\right)^2 + \left(y^{1/3}\right)^2 = 1,$$

we see that we can parameterize this by parameterizing the components as

$$x^{1/3} = \cos t \qquad \text{and} \qquad y^{1/3} = \sin t$$

for $t \in [0, 2\pi]$. The resulting path $\boldsymbol{\gamma}(t) = (\cos^3 t, \sin^3 t)$ has velocity

$$\boldsymbol{\gamma}'(t) = \left(-3\sin t\cos^2 t, 3\cos t\sin^2 t\right).$$

Taking the field $\boldsymbol{F}(x, y) = (-y, x)$ and evaluating the field along this path,

$$\boldsymbol{F}(\boldsymbol{\gamma}(t)) = \boldsymbol{F}(\cos^3 t, \sin^3 t) = (-\sin^3 t, \cos^3 t),$$

we can compute the area of the region $D$ inside of the hypocycloid as

$$\begin{aligned}
\text{area}(D) = \oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} &= \frac{1}{2} \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt \\
&= \frac{1}{2} \int_0^{2\pi} \left(-\sin^3 t, \cos^3 t\right) \cdot \left(-3\sin t \cos^2 t, 3\cos t \sin^2 t\right)\, dt \\
&= \frac{1}{2} \int_0^{2\pi} \left(3\sin^4 t \cos^2 t + 3\sin^2 t \cos^4 t\right)\, dt \\
&= \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t\, (\sin^2 t + \cos^2 t)\, dt \\
&= \frac{3}{2} \int_0^{2\pi} (\sin t \cos t)^2\, dt \\
&= \frac{3}{8} \int_0^{2\pi} \sin^2 2t\, dt \qquad \text{using} \sin 2t = 2\sin t \cos t \\
&= \frac{3\pi}{8},
\end{aligned}$$

where in the second to last line we use the trigonometric identity $\sin 2t = 2\sin t \cos t$.

### 3.3.4 Vorticity and an interpretation of Green's Theorem

Why is it that we can equate a line integral along a boundary with the two-dimensional integral of the difference of partial derivatives? To understand this, it will be important to have an an interpretation of what the quantity

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

means for a particular vector field $\boldsymbol{F}$. Consider the vector field in the region around some fixed point $(x_0, y_0)$ in the plane. For each choice of positive number $\varepsilon > 0$, we can consider the *disc of radius $\varepsilon$ centered at the point* $(x_0, y_0)$ to be the region

$$D_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 \leq \varepsilon\}.$$

We define the *vorticity* of $\boldsymbol{F}$ at the point $(x_0, y_0)$ to be the limiting value of the ratio of the circulation of $\boldsymbol{F}$ around $\partial D_\varepsilon$ to the area of $D_\varepsilon$ as $\varepsilon \to 0$,

$$\text{voriticy}\boldsymbol{F}(x_0, y_0) = \lim_{\varepsilon \to 0} \frac{1}{\text{area}(D_\varepsilon)} \oint_{\partial D_\varepsilon} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

This is essentially the *circulation of $\boldsymbol{F}$ per unit area* at the point $(x_0, y_0)$ and measures how much 'swirliness' the vector field has at a point. We can use Green's Theorem to change the line integral in the definition of the vorticity to

$$\oint_{\partial D_\varepsilon} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{D_\varepsilon} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

In the limit as $\varepsilon \to 0$, the region $D_\varepsilon$ shrinks to the point $(x_0, y_0)$ and the value of this two-dimensional integral is approximated by

$$\iint_{D_\varepsilon} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \quad \longrightarrow \quad \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bigg|_{(x_0, y_0)} \text{area}(D_\varepsilon).$$

This gives us the a useful definition for the vorticity of a vector field as a point.

**Definition 3.10.** The *vorticity* of a $C^1$ vector field $\boldsymbol{F}$ in $\mathbb{R}^2$ at a point $(x_0, y_0)$ is

$$\text{vorticity}\boldsymbol{F}(x_0, y_0) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\Bigg|_{(x_0,y_0)}$$

where $\boldsymbol{F}(x, y) = (F_1(x, y), F_2(x, y))$.

This gives us a way of rewriting Green's Theorem :

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \text{vorticity}\boldsymbol{F}\, dA$$

The line integral of $\boldsymbol{F}$ around a curve is the *circulation*, which we can interpret as how much counter clockwise rotation would result from the vector field pushing the curve around in a circle. Meanwhile, the vorticity is the amount of microscopic rotation around a single point. Therefore, Green's Theorem tells us that we can find the total amount of circulation around a curve that results from the flow of the vector field by adding up all of the microscopic bits of circulation in the area inside the the curve!

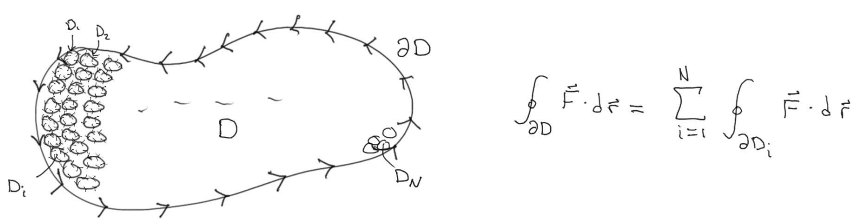



Figure 3.16: Green's Theorem tells us that we can compute the circulation around a curve by adding up all of the microscopic amounts of circulation (i.e., the vorticity) of the field at each point inside the curve.

We can get a further understanding of the vorticity by thinking about how much rotation is caused by the vector field at a single point. If we were to stick a very small paddle wheel of into the flow of the vector field at the point, how much would the paddle spin as a result of the flow at that point? The torque resulting from the vector field pushing on the edges of the paddle wheel results in the paddle wheel spinning around that point, with the magnitude of the spinning given by the vorticity of $\boldsymbol{F}$ at that point!

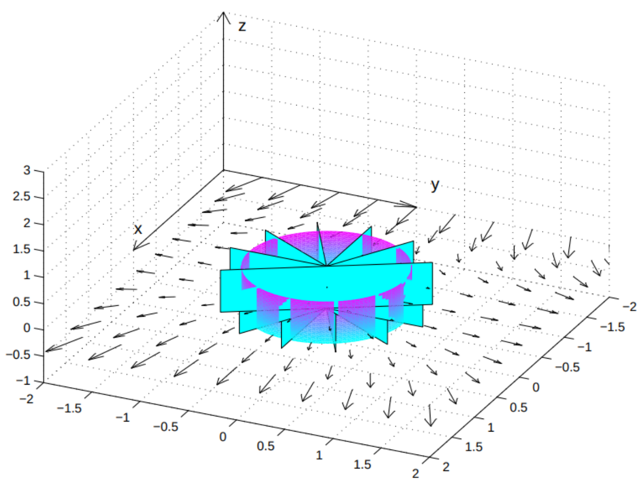



Figure 3.17: A paddle wheel inserted into the flow of a vector field at a point in the plane. The 'vorticity' tells us how much the paddle will spin counter clockwise as a result of the flow.