Lecture notes for Week 5

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5.1 Understanding parameterizations of surfaces

Now that we have spent some time parameterizing surfaces and integrating over them, we should stop for a moment to consider what it is exactly that we are doing. Modeling a surface in three dimensional space is much easier if we can view it in two-dimensional space. However, there is no way in general to completely and accurately represent a three-dimensional object on a two-dimensional plane. Consider for example the problem of printing a map of the Earth in a book. It simply can't be done without somewhere cutting and warping the surface of the globe so that it fits on a page. One way to do this is shown in Figure 5.1 below. This projection, or "flattening", of the globe corresponds to the standard *spherical coordinates* given by the parameterization $\mathbf{\Phi} : D \to \mathbb{R}$ defined as

$$\mathbf{\Phi}(\theta, \varphi) = (r \sin\varphi \cos\theta,\ r \sin\varphi \sin\theta,\ r \cos\varphi) \tag{5.1}$$

over the two-dimensional rectangle $D = [0, 2\pi] \times [0, \pi]$. This gives us a one-to-one mapping between the points on the globe and points in the rectangle, shown in Figure 5.1. Although the *points* on the sphere are in one-to-one correspondence with the points in the rectangle, smaller *regions* on the sphere and their areas get "warped" in the rectangle.

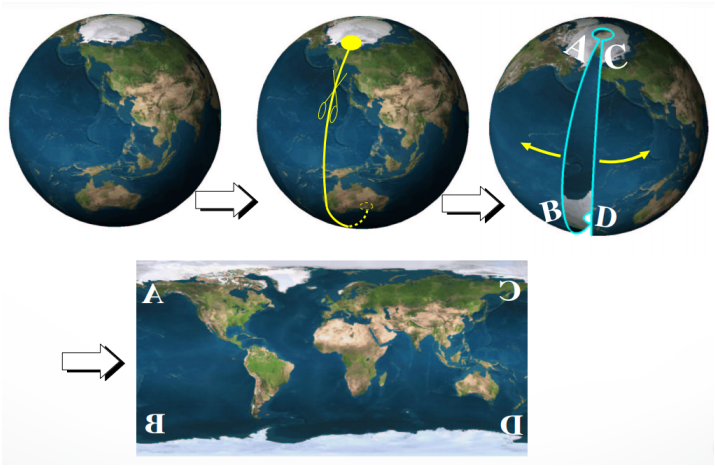

Figure 5.1: A parameterization of the sphere is a "flattening" that allows us to represent it in two-dimensional space. This projection of the globe corresponds to the spherical coordinates parameterization from last week.

The standard parameterization using spherical coordinates, however, is certainly not the only way that one can parameterize the sphere. Different parameterizations yield different 'projections' of the sphere in two-dimensional space. Figure 5.2 shows a few different examples of types of projections of the globe that are typically used in map-making.

As we have seen, a parameterization of a surface gives us a way of computing *surface integrals* on that surface. For example, given a parameterization $\mathbf{\Phi} : D \to \mathbb{R}^3$ of a surface $\Sigma \subset \mathbb{R}^3$, the area of Σ can be

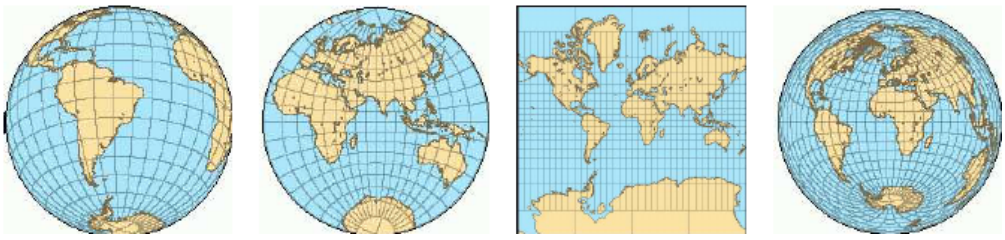

Figure 5.2: There are infinitely many ways one can parameterize the sphere to represent it as a projection in a two-dimensional plane. Here are a few typical projections used for map-making. Each parameterization 'warps' the area of a region on the sphere that it represents.

computed as

$$\text{area}(\Sigma) = \iint_\Sigma dA = \iint_D \left\| \left(\frac{\partial \mathbf{\Phi}}{\partial s} \times \frac{\partial \mathbf{\Phi}}{\partial t} \right) \Big|_{(s,t)} \right\| ds\, dt,$$

where the normal vector determined by this parameterization at a point $\mathbf{\Phi}(s,t)$ on the surface of Σ is

$$\boldsymbol{n}_{\mathbf{\Phi}}(s,t) = \left(\frac{\partial \mathbf{\Phi}}{\partial s} \times \frac{\partial \mathbf{\Phi}}{\partial t} \right) \Big|_{(s,t)}.$$

The value of $\|\boldsymbol{n}_{\mathbf{\Phi}}\|$ (which is the *magnitude* of this normal vector from this parameterization) corresponds to how much to we need "scale" the area of the piece of the region of D in order to get the correct value for the area of the corresponding piece of surface Σ. For example, if we were to use one of the parameterizations in 5.2 to compute the area of Greenland, straightforward comparison of the areas of the "projected" versions of Greenland in the projections would give us wildly different answers. Multiplying by the correct scaling factor corresponding to each parameterization gives us the correct area.

For an example of a more general surface integral that is not simply an area integral, suppose the function $f : \Sigma \to \mathbb{R}$ represents the *population density* of a point on the globe. For a parameterization $\mathbf{\Phi}$, the total population on the Earth can be computed as

$$\text{total population} = \iint_\Sigma f\, dA = \iint_D f(\mathbf{\Phi}(s,t)) \left\| \left(\frac{\partial \mathbf{\Phi}}{\partial s} \times \frac{\partial \mathbf{\Phi}}{\partial t} \right) \Big|_{(s,t)} \right\| ds\, dt. \tag{5.2}$$

If we forget to include the area-scaling factor $\|\boldsymbol{n}_{\mathbf{\Phi}}(s,t)\|$ inside the integral in (5.2), we would get the wrong answer! For example, we would most likely get an overestimate for the population of Greenland and underestimate the population countries closer to the equator.

5.2 Change of variables in two dimensions and Polar coordinates

Recall from first-semester calculus that we sometimes use a "change of variables" (or "u-substitution") to simplify complicated integrals into expressions that are easier to manage. Namely, if u is a continuous one-to-one function on some interval $[a,b]$, the change of variables formula tells us that

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du. \tag{5.3}$$

The integral on the left is over the interval $[a,b]$. The function u "transforms" this interval to a new interval $[u(a), u(b)]$ and the integral on the right integrates the new variable over this transformed interval. We can do something similar with integrals in two-dimensions.

5.2.1 Jacobian of a transformation

Our formulas for computing surface integrals of two-dimensional surfaces living in three-dimensional space also has another application: we can use it to make use of *reparameterizations* of two-dimensional regions by making a *change of variables*. When considering transformations like this, it is customary to use u and v as the variable names. Consider for example a one-to-one mapping $\mathbf{\Phi} : D \to \mathbb{R}^2$ from a region $D \subseteq \mathbb{R}^2$ in the plane, as shown in Figure 5.3, which has components

$$\mathbf{\Phi}(u, v) = (x(u, v),\, y(u, v)). \tag{5.4}$$

The new region $\mathbf{\Phi}(D) \subseteq \mathbb{R}^2$ that is parameterized by $\mathbf{\Phi}$ is the set of points

$$\mathbf{\Phi}(D) = \big\{ \mathbf{\Phi}(u, v) \,\big|\, (u, v) \in D \big\}$$

that gets mapped to by $\mathbf{\Phi}$.

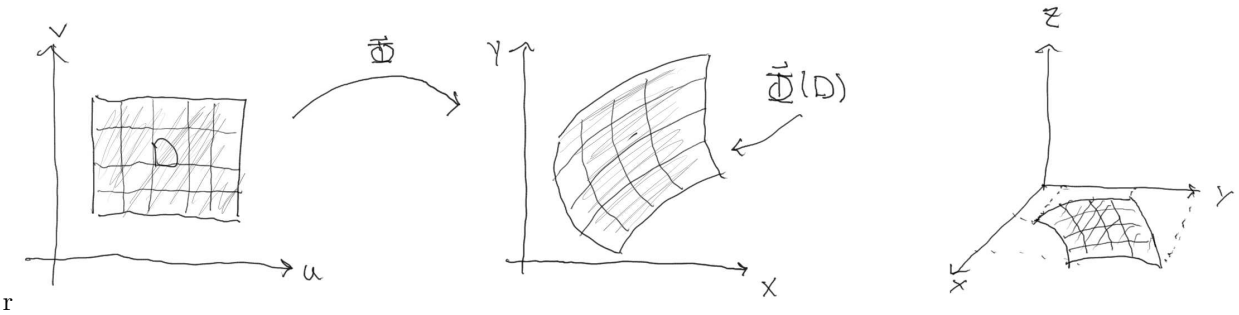

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Figure 5.3: A one-to-one mapping $\mathbf{\Phi} : D \to \mathbb{R}^2$ transforms a region $D \subseteq \mathbb{R}^2$ into another region $\mathbf{\Phi}(D)$ in the plane by "reparameterizing" the region. In three dimensions, it can be viewed as sitting on the xy-plane in space.

We may view this region $\mathbf{\Phi}(D)$ as a surface Σ "in the xy-plane" of three-dimensional space

$$\Sigma = \{(x(u, v), y(u, v), 0) \,|\, (u, v) \in D\},$$

which we can think of as being a parameterized surface that is parameterized by the same parameterization in (5.4) and adding a zero in the z-coordinate:

$$\mathbf{\Phi}(u, v) = (x(u, v),\, y(u, v),\, 0)$$

to get the parameterization of the corresponding surface n $\mathbb{R}^3$.

The "normal vector" of Σ determined by this parameterization can be found by computing the partial derivatives

$$\frac{\partial \mathbf{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right) \qquad \text{and} \qquad \frac{\partial \mathbf{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right),$$

and taking the cross product

$$\boldsymbol{n}_{\mathbf{\Phi}} = \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \hat{\boldsymbol{k}} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \hat{\boldsymbol{k}}. \tag{5.5}$$

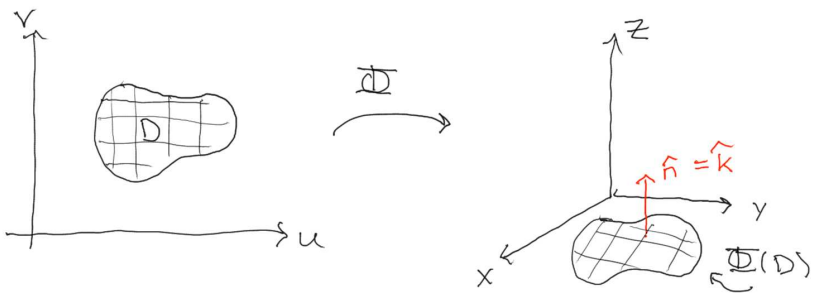


Figure 5.4: The normal vector of the surface on the xy-plane points along the z-axis.

The determinant in (5.5) is known as the *Jacobian* of the transformation. We commonly use either the notation $J_{\boldsymbol{\Phi}}$ or $\dfrac{\partial(x,y)}{\partial(u,v)}$ to denote the Jacobian of a transformation:

$$\boxed{J_{\boldsymbol{\Phi}} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u},} \tag{5.6}$$

such that the magnitude of the normal vector of the transformation at the point $\boldsymbol{\Phi}(u,v)$ is

$$\|\boldsymbol{n}_{\boldsymbol{\Phi}}(u,v)\| = |J_{\boldsymbol{\Phi}}(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|.$$

The Jacobian is therefore a function $J_{\boldsymbol{\Phi}} : D \to \mathbb{R}$ that takes points $(u,v) \in D$ and maps them to values

$$J_{\boldsymbol{\Phi}}(u,v) = \left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u,v)}.$$

This value corresponds to how much the area around the point (u,v) gets "stretched" under the transformation $\boldsymbol{\Phi}$ from the uv-plane to the xy-plane.

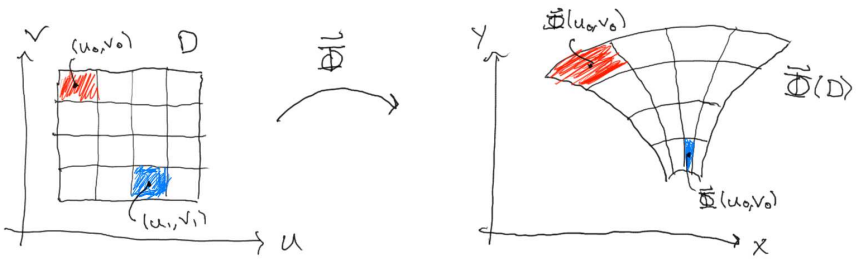


Figure 5.5: A transformation "stretches" different parts of the region D by different amounts. The Jacobian of a transformation at a point (u_0, v_0) is the amount of stretching in the region around that point.

We are now ready to present a "change of variables" formula for two-dimensional integrals. First recall our formula for surface integrals from last week. For a parameterization $\mathbf{\Phi} : D \to \mathbb{R}^3$ of a surface $\Sigma \subseteq \mathbb{R}^3$ and a continuous scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, the integral of f over the surface Σ is computed by

$$\iint_\Sigma f \, dA = \iint_D f(\mathbf{\Phi}(s,t)) \left\| \frac{\partial \mathbf{\Phi}}{\partial s} \times \frac{\partial \mathbf{\Phi}}{\partial t} \right\| ds \, dt,$$

where in the integral on the right we integrate over the variables s and t.

For a reparameterization $\mathbf{\Phi} : D \to \mathbb{R}^2$ of a region $D \subseteq \mathbb{R}^2$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, the integral of f over $\mathbf{\Phi}(D)$ is computed by making a change of variables:

$$\begin{aligned} \iint_{\mathbf{\Phi}(D)} f(x,y) \, dx \, dy &= \iint_D f\big(x(u,v), y(u,v)\big) \left| J_{\mathbf{\Phi}}(u,v) \right| du \, dv \\ &= \iint_D f\big(\mathbf{\Phi}(u,v)\big) \left| J_{\mathbf{\Phi}}(u,v) \right| du \, dv \\ &= \iint_D f\big(\mathbf{\Phi}(u,v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv \end{aligned} \tag{5.7}$$

5.2.2 Polar coordinates

One of the most useful examples of a coordinate transformation in the plane is the use of *polar coordinates*:

$$\mathbf{\Phi}(r,\theta) = (x(r,\theta), y(r,\theta))$$

where

$$x(r,\theta) = r\cos\theta \text{ and } y(r,\theta) = r\sin\theta \qquad \text{for } 0 \le \theta < 2\pi \text{ and } 0 \le r. \tag{5.8}$$

This is a reparameterization of the plane $\mathbb{R}^2$. Each point is specified by its distance r from the origin and the angle θ that the point makes with the x-axis.

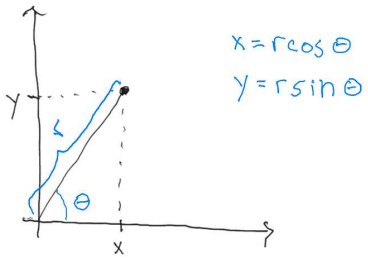


Figure 5.6: Polar coordinates are a way or reparameterizing $\mathbb{R}^2$. The variables $0 \le \theta < 2\pi$ and $0 \le r$ parameterize the plane by $x = r\cos\theta$ and $y = r\sin\theta$.

Given a region $D \in \mathbb{R}^2$ in the xy-plane, we are often interested in finding the region in the $r\theta$-plane that gets mapped to D under the polar coordinate transformation. We denote this set as

$$\mathbf{\Phi}^{-1}(D) = \big\{ (r,\theta) \,\big|\, \mathbf{\Phi}(r,\theta) \in D \big\}.$$

Example 5.1. The polar coordinate transformation is useful for converting "circular" regions in the xy-plane into rectangles in the $r\theta$-plane, which are easier to integrate over. Consider for example the following regions.

- Let D_1 be the "quarter disc" that is inside the circle $x^2 + y^2 = 1$ and in the quadrant where $x, y \geq 0$. In polar coordinates, this region is a rectangle

$$\mathbf{\Phi}^{-1}(D_1) = [0, 1] \times [0, \pi/2]$$

of the points where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$.

- Let D_2 be the annulus that is inside the circle $x^2 + y^2 = 4$ and outside the circle $x^2 + y^2 = 1$. In polar coordinates, this region is a rectangle

$$\mathbf{\Phi}^{-1}(D) = [1, 2] \times [0, 2\pi]$$

of the points where $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

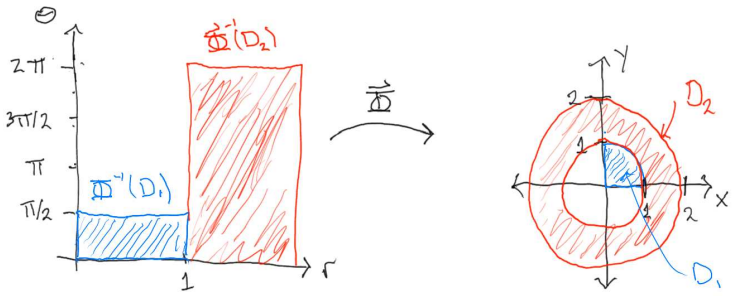


Figure 5.7: The regions D_1 and D_2 in the xy-plane from Example 5.1 and the regions $\mathbf{\Phi}^{-1}(D_1)$ and $\mathbf{\Phi}^{-1}(D_2)$ in the $r\theta$-plane that get mapped to those regions under the polar coordinates transformation.

We can use polar coordinates to simplify some two-dimensional integrals using the formula in (5.7). The partial derivatives of the transformation in (5.8) for polar coordinates are

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

which gives us the Jacobian for the polar coordinates transformation:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r.$$

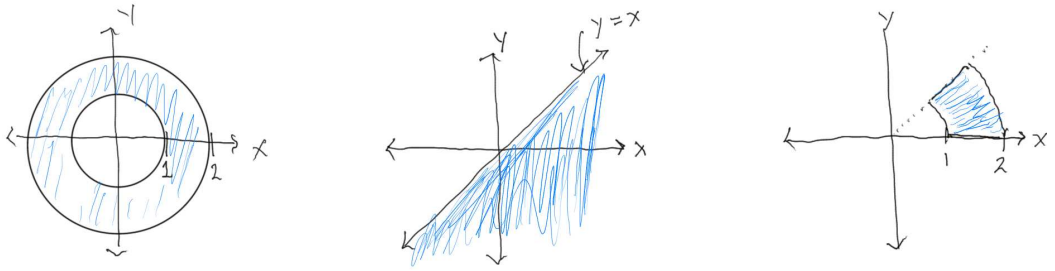
This gives us the general formula for computing a two-dimensional integral using polar coordinates:

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{\mathbf{\Phi}^{-1}(D)} f((x(r,\theta), y(r,\theta)))\,r\,dr\,d\theta}.$$

Example 5.2. Evaluate the integral $\iint_D xy\,dx\,dy$ where D is part of an annulus in the first quadrant of plane in the region where $y \leq x$:

$$D = \left\{(x,y) \,\middle|\, 1 \leq x^2 + y^2 \leq 4,\ x, y \geq 0,\ \text{and } y \leq x\right\}.$$

Since we have a constraint involving an expression of the form $x^2 + y^2$, it is likely that using polar coordinates will simplify the integral. As always, it will first be helpful to sketch the region to get an idea of what it



(a) The annulus $1 \leq x^2 + y^2 \leq 2$. (b) The half-plane where $y \leq x$. (c) The region D from example 5.2.

Figure 5.8: Sketching the region D in Example 5.2

looks like. The region where $1 \leq x^2 + y^2 \leq 2$ is an annulus shown in Figure 5.8a, while the region where $y \leq x$ is shown in Figure 5.8b. The region D is the intersection of these regions is shown in Figure 5.8c.

From this sketch, we see that the region D can be parameterized using polar coordinates where the parameter for the radius varies over the range $1 \leq r \leq 2$ while the parameter for the angle will be taken between $0 \leq \theta \leq \pi/4$. Changing to polar coordinates, the integral becomes

$$\begin{aligned}
\iint_D xy\,dx\,dy &= \int_0^{\frac{\pi}{4}} \int_1^2 \left(r^2 \cos\theta \sin\theta\right) r\,dr\,d\theta \\
&= \left(\int_0^{\frac{\pi}{4}} \cos\theta \sin\theta\,d\theta\right)\left(\int_1^2 r^3\,dr\right) \\
&= \left(\frac{1}{2}\sin^2\theta\Big|_0^{\pi/4}\right)\left(\frac{1}{4}r^4\Big|_1^2\right) \\
&= \left(\frac{1}{4}\right)\left(\frac{15}{4}\right) = \frac{15}{16}.
\end{aligned}$$

5.3 Surface integrals of vector fields

In Week 4 we obtained a formula for the surface integral of a scalar field f over a surface Σ with a parametric representation $\mathbf{\Phi} : D \to \mathbb{R}^3$. In this section, our goal is to define an integral of a *vector field* over a surface, analogous to how we defined line integrals of vector fields over curves. Before proceeding, we must first introduce the concept of an *orientation* on a surface.

5.3.1 Oriented surfaces and boundaries

Definition 5.3. An *orientation* on a C^1 surface $\Sigma \subseteq \mathbb{R}^3$ is a way of continuously assigning a unit normal vector to each point on the surface. A surface together with an orientation is called an *oriented surface*.

If a surface can be given an orientation, it is said to be *orientable*. Just as oriented *curves* could have one of exactly two opposite orientations, any orientable surface will have exactly two possible orientations. Giving an orientation to a surface is simply deciding which "side" of the surface we want to denote as "up" (or "outward") and which side is "down" (or "inward").

Although it will not be important for this course, it is important to note that some surfaces *cannot* be given an orientation. The *Möbius strip* is an example of a surface which is not orientable, since there it only has one "side." There is no continuous way to assign a normal unit vector at each point.

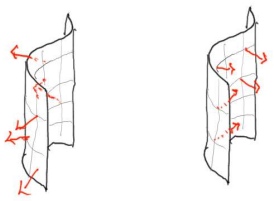


Figure 5.9: A surface can have one of two possible orientations.

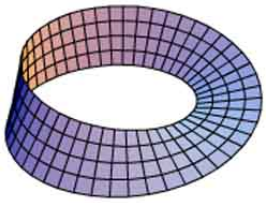


Figure 5.10: The Möbius strip is an example of a non-orientable surface, since it only has one "side."

If a smooth surface Σ has a parameterization $\boldsymbol{\Phi} : D \to \mathbb{R}^3$, that parameterization automatically provides the surface with an orientation. The unit normal vector at the point $\boldsymbol{\Phi}(s,t) \in \Sigma$ that we obtain from this parameterization is

$$\hat{\boldsymbol{n}}(\boldsymbol{\Phi}(s,t)) = \frac{\boldsymbol{n}_{\boldsymbol{\Phi}}(s,t)}{\|\boldsymbol{n}_{\boldsymbol{\Phi}}(s,t)\|} = \frac{\dfrac{\partial \boldsymbol{\Phi}}{\partial s} \times \dfrac{\partial \boldsymbol{\Phi}}{\partial t}}{\left\| \dfrac{\partial \boldsymbol{\Phi}}{\partial s} \times \dfrac{\partial \boldsymbol{\Phi}}{\partial t} \right\|}.$$

If two different parameterizations always give the same unit normal vector, then they give the same orientation to the surface. If a surface is given with an orientation, we must be careful to choose a parameterization that gives the normal vector pointing in the correct direction.

Recall our notion of the "boundary" ∂D of a region $D \subseteq \mathbb{R}^2$, which is a closed oriented curve. Any oriented surface in space likewise a boundary. Given a smooth oriented surface $\Sigma \subseteq \mathbb{R}^3$, its *boundary* is the closed oriented curve denoted by

$$\boxed{\text{boundary of } \Sigma = \partial\Sigma}$$

whose orientation is always given by the *right-hand rule*. At a point on the boundary of the surface, the orientation of the boundary at that point can be found by pointing your thumb of your right hand in the direction of the normal vector at that point and lining up your pinky along the boundary at that point.

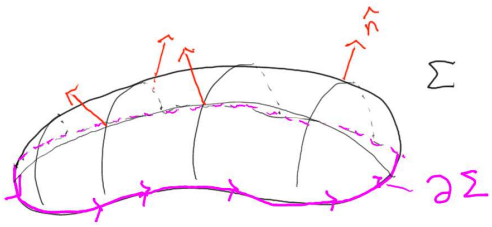


Figure 5.11: The boundary of an oriented surface is an oriented curve. The orientation of the boundary is found using the right-hand rule.

Your fingers point in the direction of the boundary's orientation at that point.

5.3.2 Flux through a surface

Before digging into the details of surface integration, we first introduce the idea of the "flux" of a vector field through a window. Suppose that $\boldsymbol{F}$ is a constant vector field in space that describes, perhaps, the velocity of water flowing through space. Given a small window with area ΔA inside the flow of the water, what is the *total* rate at which water flows through the window (in units of volume per unit time)? This will depend on the *angle* that the window makes with the direction of the flow. For an arbitrary field $\boldsymbol{F}$, this quantity is called the *flux* of the field through the window. If the normal of the window is parallel with the flow of the light, the total amount of water flowing through the window will be proportional to

$$\text{total flux} = \|\boldsymbol{F}\|\Delta A.$$

However, if the normal of the window is at an *angle* with the direction of the flow, the total flux will depend on that angle:

$$\text{total flux} = \|\boldsymbol{F}\|\Delta A\cos\theta.$$

If the unit normal vector of the small window is denoted $\hat{\boldsymbol{n}}$, this total flux can be denoted using the dot product of these vectors by

$$\text{total flux} = \|\boldsymbol{F}\|\Delta A\cos\theta = \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\Delta A.$$

For a small flat piece of surface in space with a specified orientation, it is common to denote this piece of surface as the vector $\Delta\boldsymbol{A} = \Delta A\,\hat{\boldsymbol{n}}$ that combines the magnitude of the area ΔA with the direction of the orientation of that area $\hat{\boldsymbol{n}}$ into a single vector, such that the flux through the window is equal to $\boldsymbol{F}\cdot\Delta\boldsymbol{A}$.

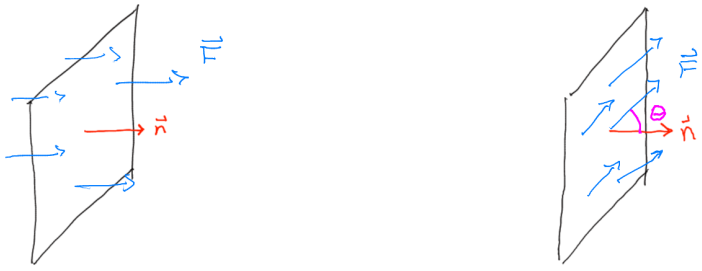


Figure 5.12: Flux of a constant vector field through a flat window. The total flow through the window depends on the angle between the field and the normal vector.

5.3.3 Flux integrals

We are now ready to introduce the idea of a surface integral over a vector field. We proceed exactly as we did in our introduction of surface integrals of *scalar* fields. Let Σ be an oriented surface with a specified orientation that is parameterized by $\boldsymbol{\Phi} : D \to \mathbb{R}^3$, where we may suppose for simplicity that $D \subseteq \mathbb{R}^2$ is a rectangle

$$D = \big\{(u,v)\,\big|\,a \le s \le b \text{ and } c \le t \le d\big\} = [a,b]\times[c,d].$$

Suppose we are also given a continuous vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and we are asked to find the *total flux* of the field $\boldsymbol{F}$ that flows through the surface Σ. To do this, we need to take the dot product of the field $\boldsymbol{F}$ with the *unit normal vector* $\hat{\boldsymbol{n}}$ at each point on the surface, and integrate this value over the entire surface.

We can approximate the value of this integral by segmenting the rectangle D into a grid of N smaller rectangles $D_1,\ldots,D_N$ (each of which has sides of lengths Δs and Δt) also segments the surface into smaller

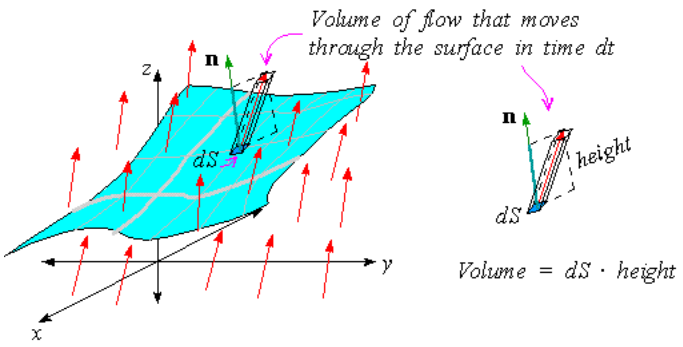


Figure 5.13: The flux of a field $\boldsymbol{F}$ through an oriented surface Σ is equal to the integral of the dot product $\boldsymbol{F} \cdot \hat{\boldsymbol{n}}$ of the field with the unit normal vector at each point on the surface.

pieces of surface $\Sigma_1, \ldots, \Sigma_N$ (each of which can be approximated as a parallelogram with sides given by the vectors

$$\left.\frac{\partial \boldsymbol{\Phi}}{\partial s}\right|_{(s_i,t_i)} \Delta s \qquad \text{and} \qquad \left.\frac{\partial \boldsymbol{\Phi}}{\partial t}\right|_{(s_i,t_i)} \Delta t,$$

where (s_i, t_i) is a point in the region D_i). As we determined in our analysis of integrals of *scalar* fields over surfaces, the area of each piece Σ_i is approximately

$$\Delta A_i = \text{area}(\Sigma_i) \approx \left\| \left(\frac{\partial \boldsymbol{\Phi}}{\partial s} \times \frac{\partial \boldsymbol{\Phi}}{\partial t} \right) \Big|_{(s_i,t_i)} \right\| \Delta s \Delta t = \|\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\| \Delta s \Delta t,$$

where $\boldsymbol{n}_{\boldsymbol{\Phi}}$ is the (unnormalized) normal vector at that point on the surface determined by this parameterization. Meanwhile, the *unit* normal vector at this point is

$$\hat{\boldsymbol{n}}\big(\boldsymbol{\Phi}(s_i, t_i)\big) = \frac{\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)}{\|\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\|}.$$

The amount of flux through the tine piece of the surface Σ_i can therefore be approximated by

$$\begin{aligned} \text{flux through } \Sigma_i &\approx \boldsymbol{F}(\boldsymbol{\Phi}(s_i, t_i)) \cdot \hat{\boldsymbol{n}}\big(\boldsymbol{\Phi}(s_i, t_i)\big)\, \text{area}(\Sigma_i) \\ &= \boldsymbol{F}(\boldsymbol{\Phi}(s_i, t_i)) \cdot \left(\frac{\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)}{\|\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\|} \right) \big(\|\boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\| \Delta s \Delta t \big) \\ &= \boldsymbol{F}(\boldsymbol{\Phi}(s_i, t_i)) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\, \Delta s \Delta t. \end{aligned}$$

The total flux through the entire surface can therefore be approximated by summing up the flux through each of the pieces,

$$\text{total flux through } \Sigma = \sum_{i=1}^{N} \boldsymbol{F}(\boldsymbol{\Phi}(s_i, t_i)) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(s_i, t_i)\, \Delta s \Delta t.$$

Taking the limit of this value as $N \to \infty$ and $\Delta s, \Delta t \to 0$, where we segment the region D into infinitely many pieces of infinitesimally small area, we get the integral

$$\text{total flux through } \Sigma = \iint_D \boldsymbol{F}(\boldsymbol{\Phi}(s, t)) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(s, t)\, ds\, dt.$$

As with all of the other integrals we have defined in this course, it does not matter which parameterization $\boldsymbol{\Phi}$ that we choose to parameterize Σ (as long as the parameterization provides the correct orientation), so we use the notation

$$\boxed{\iint_\Sigma \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA = \iint_D \boldsymbol{F}(\boldsymbol{\Phi}(s, t)) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(s, t)\, ds\, dt.}$$

to denote the value of this integral. This is the *surface integral of a vector field* $\boldsymbol{F}$ *through an oriented surface* Σ. We also sometimes write this integral using the notation

$$\iint_\Sigma \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA = \iint_\Sigma \boldsymbol{F} \cdot d\boldsymbol{A}$$

where $d\boldsymbol{A} = \hat{\boldsymbol{n}}\, dA$ is the differential piece of oriented surface.

When solving problems involving flux integrals, it is recommended that you always sketch the surface to get an idea of what the situation looks like. This can provide intuition toward solving the problem and it will help you verify that you have the correct orientation for your parameterization.

Example 5.4. Water flows through an oriented surface Σ that is determined by the equation $y = x^2$ in the region where $0 \le x \le 2$ and $0 \le z \le 3$ and whose orientation points outward toward the x-axis. If the velocity of the water flow at a given point in space is given by the vector field

$$\boldsymbol{F}(x, y, z) = (3z^2, 6, 6xz),$$

compute the rate at which water flows through the surface Σ. It will be helpful to first parameterize and sketch the surface before to get a physical intuition of the problem and to make sure our parameterization provides the correct orientation. Choosing the parameterizations for the x- and z-coordinates as

$$x = s \text{ for } 0 \le s \le 2 \quad \text{and} \quad z = t \text{ for } 0 \le t \le 3,$$

we may parameterize the surface using $\boldsymbol{\Phi} : D \to \mathbb{R}^3$ as

$$\boldsymbol{\Phi}(s, t) = (s, s^2, t)$$

where $D = [0, 2] \times [0, 3]$. The resulting surface is simply the curve of the graph of the function $y = x^2$ on the xy-plane that is extruded upward along the z-axis (see Figure 5.14).

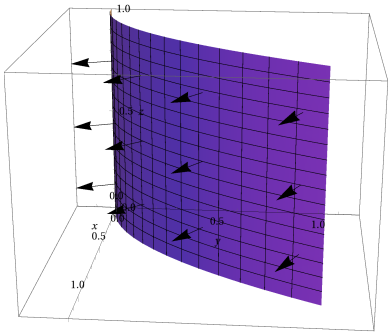


Figure 5.14: The parabolic surface Σ from Example 5.4 with the specified orientation (outward toward the x-axis) indicated.

To compute the desired flux integral we must determine the normal vector at each point on the surface. The tangent vectors are

$$\frac{\partial \boldsymbol{\Phi}}{\partial s} = (1, 2s, 0) \qquad \text{and} \qquad \frac{\partial \boldsymbol{\Phi}}{\partial t} = (0, 0, 1),$$

and the normal vector at the point $\boldsymbol{\Phi}(s, t)$ from this parameterization is

$$\begin{aligned} \boldsymbol{n}_{\boldsymbol{\Phi}}(s, t) &= (1, 2s, 0) \times (0, 0, 1) \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 2s & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2s\, \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} = (2s, -1, 0). \end{aligned}$$

Before proceeding, we must first verify that this normal vector gives us the correct orientation. At the point $(s,t) = (1,1)$ in the region D, the corresponding point on the surface is $\boldsymbol{\Phi}(1,1) = (1,1,1)$ and the normal vector we found at that point is $\boldsymbol{n}_{\boldsymbol{\Phi}}(1,1) = (2,-1,0)$. This is indeed the vector pointing in the "outward" direction of the parabolic surface and towards the x-axis in the figure, so our parameterization gave us the correct orientation. (If the parameterization had given us the opposite orientation, we would have needed to multiply the value of the resulting integral by -1 to correct for the direction.)

The field at each point on the surface is $\boldsymbol{F}(\boldsymbol{\Phi}(s,t)) = (3t^2, 6, 6st)$. The total flux is therefore given by

$$\begin{aligned}\iint_\Sigma \boldsymbol{F}\cdot d\boldsymbol{A} &= \iint_D \boldsymbol{F}(\boldsymbol{\Phi}(s,t))\cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(s,t)\,ds\,dt \\ &= \iint_D (3t^2,6,6st)\cdot(2s,-1,0)\,ds\,dt \\ &= \int_0^3 \left(\int_0^2 (6st^2-6)\,ds\right) dt \\ &= \cdots \\ &= 72.\end{aligned}$$

In "real life" we would be required to provide units with our answers. For example, if we suppose that the velocity field is given in units of $\mathrm{m}\cdot\mathrm{s}^{-1}$, integrating this over an area would give us a value in units of $\mathrm{m}^3\cdot\mathrm{s}^{-1}$ for the total flow rate that we compute.

5.4 Curl and Stokes' Theorem

5.4.1 Curl

Recall that the *vorticity* of a C^1-vector field on $\mathbb{R}^2$ at a point (x,y) is defined as

$$(\operatorname{vor}\boldsymbol{F})(x,y) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bigg|_{(x,y)}.$$

It describes how much the vector field 'spins' at the point (x,y). That is, if we stick a very small paddlewheel in the flow at that point, the vorticity tells how much angular velocity that paddlewheel picks up as a result of the flow. The analogous vector field in three dimensions is the *curl*.

Definition 5.5. Let $\boldsymbol{F} : D \to \mathbb{R}^3$ be a C^1-vector field region $D \subseteq \mathbb{R}^3$ in space with components

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\hat{\boldsymbol{\imath}} + F_2(x,y,z)\hat{\boldsymbol{\jmath}} + F_3(x,y,z)\hat{\boldsymbol{k}}. \tag{5.9}$$

The *curl* of the vector field is the vector field $\operatorname{curl}\boldsymbol{F}$ on the same region D defined by

$$(\operatorname{curl}\boldsymbol{F})(x,y,z) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bigg|_{(x,y,z)}\hat{\boldsymbol{\imath}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\bigg|_{(x,y,z)}\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bigg|_{(x,y,z)}\hat{\boldsymbol{k}}$$

for all (x,y,z) in D.

Remark. The definition of the curl in (5.9) looks like a confusing jumble of symbols. Just as we defined the *gradient* of a scalar field using the symbol ∇ as

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

The gradient of a scalar field is a vector field, and we can think of ∇ as an "operator" that is also a vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}.$$

Using this notation, we can write the curl of a vector field as a *cross product* of the ∇-operator with the $\boldsymbol{F}$ as

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Recall that the *vorticity* of a vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ tells us what the "local spin" in the flow is at a point in the plane. That is, how fast a small paddlewheel placed at that point would spin.

In three dimensions the curl has a similar interpretation, but the resulting spin now depends on what "angle" the paddlewheel sits. Figure 5.15 shows a small paddle wheel placed at the point (x, y, z) in space and whose axis of rotation is along the vector $\hat{\boldsymbol{n}}$. If we allow the paddlewheel to rotate the flow of the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ at that point, then the value of the dot product $(\nabla \times \boldsymbol{F}(x, y, z)) \cdot \hat{\boldsymbol{n}}$ tells us exactly how much (and in which direction) spinning would result from the flow. The axis of roation that would result in the *greatest* amount of angular velocity in the spinning paddlewheel is the direction of the curl $\nabla \times \boldsymbol{F}$ at that point.

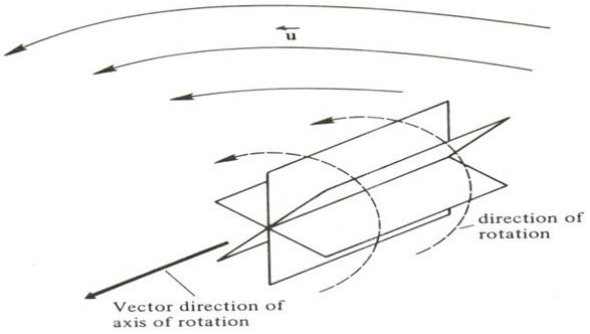


Figure 5.15: The curl can be given a physical meaning. Place a small paddlewheel at a point in the field and let the flow of the vector field cause the paddlewheel to spin. The maximum angular velocity of the paddlewheel at that point is proportional to the magnitude of the curl, while the axis of maximum angular momentum is the direction of the curl.

5.4.2 Stokes' Theorem

The curl of a vector field in $\mathbb{R}^3$ is the generalization of the vorticity of a vector field in $\mathbb{R}^2$. In the plane, recall that Green's Theorem equates line integrals of a vector field along a boundary of a region with the two-dimensional integral of the vorticity:

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D (\operatorname{vor} \boldsymbol{F})\, dA.$$

Stokes' Theorem is essentially a generalization of Green's Theorem.

Theorem 5.6 (Stokes' Theorem)**.** *Let $\Sigma \in \mathbb{R}^3$ be a connected oriented surface and let $\boldsymbol{F}$ be a vector field that is C^1 on Σ. Then*

$$\oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{\Sigma} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A}.$$

Since we have already proved Green's Theorem we will not bother presenting the details of a proof of Stokes' Theorem. However, we will note that the two theorems have essentially the same physical interpretation. That is, computing the *circulation* of a vector field around a closed curve is equivalent to adding up all of the microscopic amounts of circulation around all of the points inside that curve.

Remark. Recall that the line integral of a gradient field is independent of path. That is, if Γ_0 and Γ_1 are two oriented C^1-curves with the same start and end points, then

$$\int_{\Gamma_0} \nabla\Psi \cdot d\boldsymbol{r} = \int_{\Gamma_1} \nabla\Psi \cdot d\boldsymbol{r} = \Psi(\boldsymbol{\gamma}(b)) - \Psi(\boldsymbol{\gamma}(a))$$

where $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^3$ is a parameterization of either Γ_0 or Γ_1. That is, "the line integral of gradient field is independent of path." Stokes' Theorem gives us an analogous statement for flux integrals over *surfaces* with the same *boundary*. If Σ_0 and Σ_1 are two smooth oriented surfaces with the same boundary $\partial\Sigma_0 = \partial\Sigma_1$, then

$$\iint_{\Sigma_0} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A} = \oint_{\partial\Sigma_0} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\partial\Sigma_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{\Sigma_1} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A}.$$

That is, "the flux integral of a curl is independent of surface." This fact will have many useful applications in electromagnetism that we will see later on.

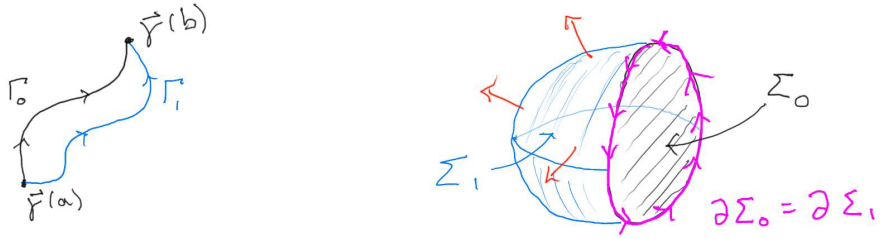


Figure 5.16: Two oriented curves with the same start and end points, and two oriented surfaces with the same boundary.

Example 5.7. Consider a vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined for all $(x, y, z) \in \mathbb{R}^3$ as

$$\boldsymbol{F}(x, y, z) = x\,\hat{\boldsymbol{\imath}} + (x + y)\hat{\boldsymbol{\jmath}} + (x + y + z)\hat{\boldsymbol{k}}$$

and let Σ be the top half of the sphere with radius $r > 0$ centered at the origin with outward orientation. We want to calculate the surface integral of the curl field $\nabla \times \boldsymbol{F}$ through Σ. Note that the boundary of Σ is the circle of radius r on the xy-plane and centered at the origin (and oriented counterclockwise when viewed from above). See Figure 5.17.

We could evaluate this integral using Stokes' Theorem to turn the surface integral into a line integral along the boundary, but this line integral turns out to be quite tedious. However, we may see from 5.17 that the *flat surface*

$$\Sigma_1 = \left\{(x, y, z) \,\middle|\, z^2 + y^2 \le r^2 \text{ and } z = 0\right\},$$

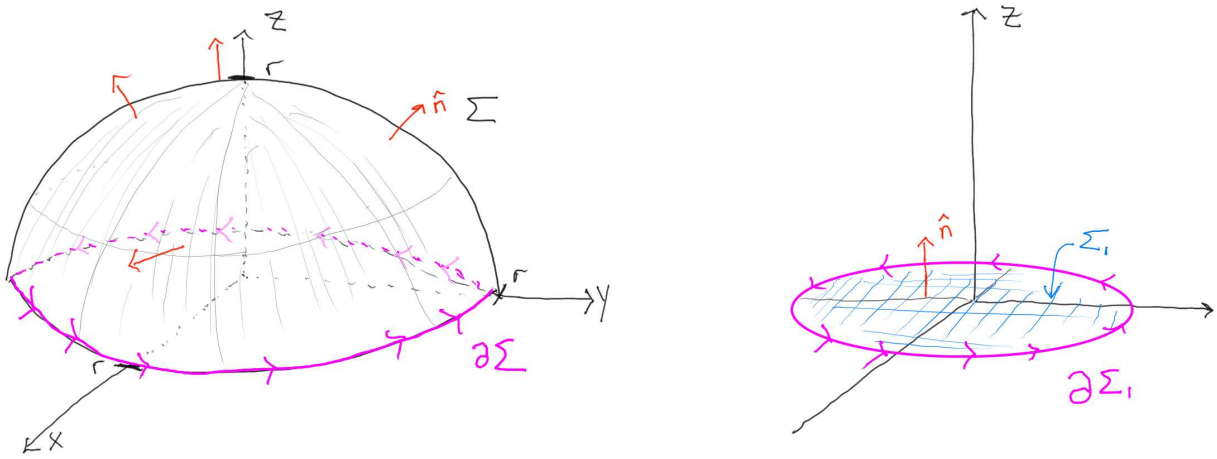


Figure 5.17: The boundary of the top half of the sphere is the same as the boundary of the disc on the xy-plane.

which is the disc of radius r on the xy-plane centered at the origin, has the *same boundary* as Σ, where we take the upward orientation of this surface with normal vector $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$ everywhere on this surface. From the previous remark, we get

$$\iint_{\Sigma} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A} = \iint_{\Sigma_1} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A},$$

where we can give Σ_1 the very simple parameterization $\boldsymbol{\Phi} : D \to \mathbb{R}^3$ defined as

$$\boldsymbol{\Phi}(s,t) = s\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$

for all $(s,t) \in D$ where $D \subseteq \mathbb{R}^2$ is the disc in plane

$$D = \{(s,t) \,|\, s^2 + t^2 \leq r^2\}.$$

The partial derivatives of this parameterization are simply

$$\left.\frac{\partial \boldsymbol{\Phi}}{\partial s}\right|_{(s,t)} = \hat{\boldsymbol{\imath}} \qquad \text{and} \qquad \left.\frac{\partial \boldsymbol{\Phi}}{\partial t}\right|_{(s,t)} = \hat{\boldsymbol{\jmath}}$$

for all $(s,t) \in D$ from which we get

$$\left.\left(\frac{\partial \boldsymbol{\Phi}}{\partial s} \times \frac{\partial \boldsymbol{\Phi}}{\partial s}\right)\right|_{(s,t)} = \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$$

for all $(s,t) \in D$. From the definition in 5.5, the curl of this vector field is

$$(\nabla \times \boldsymbol{F})(x,y,z) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$$

for all (x,y,z), which we see is a constant vector field, and thus the curl at each point on the surface Σ_1 is given by

$$(\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(s,t)) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$$

for all $(s,t) \in D$. The unit normal vector everywhere on the surface Σ_1 is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, so

$$(\nabla \times \boldsymbol{F}) \cdot \left(\left.\left(\frac{\partial \boldsymbol{\Phi}}{\partial s} \times \frac{\partial \boldsymbol{\Phi}}{\partial s}\right)\right|_{(s,t)} \right) = (\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{k}} = 1,$$

and we obtain

$$\iint_{\Sigma_1} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A} = \iint_D (\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(s,t)) \cdot \left(\left(\frac{\partial \boldsymbol{\Phi}}{\partial s} \times \frac{\partial \boldsymbol{\Phi}}{\partial s} \right) \Big|_{(s,t)} \right) ds\, dt$$
$$= \iint_D dA = \text{area}(D) = \pi r^2.$$

5.4.3 Irrotational fields

Definition 5.8. A C^1-vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is said to be *irrotational* if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ on all of Ω.

Recall that a *conservative field* is a vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ that is just the gradient of some scalar field $\Psi : \Omega \to \mathbb{R}^3$ (i.e., $\nabla \Psi = \boldsymbol{F}$). The curl of a conservative C^2-scalar field must be zero, since

$$\nabla \times (\nabla \Psi) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Psi}{\partial x} & \frac{\partial \Psi}{\partial y} & \frac{\partial \Psi}{\partial z} \end{vmatrix}$$
$$= \underbrace{\left(\frac{\partial^2 \Psi}{\partial y \partial z} - \frac{\partial^2 \Psi}{\partial z \partial y} \right)}_{=0} \hat{\boldsymbol{\imath}} - \underbrace{\left(\frac{\partial^2 \Psi}{\partial x \partial z} - \frac{\partial^2 \Psi}{\partial z \partial x} \right)}_{=0} \hat{\boldsymbol{\jmath}} + \underbrace{\left(\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} \right)}_{=0} \hat{\boldsymbol{k}} = 0, \tag{5.10}$$

where we note that it does not matter the order of the partial derivatives of Ψ in (5.10) since Ψ is C^2. In particular, we see that conservative fields are irrotational.

Theorem 5.9. *Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected region in space and let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a C^1-vector field. The following are equivalent:*

1. *The field $\boldsymbol{F}$ is conservative.*
2. *The field $\boldsymbol{F}$ is irrotational (i.e. $\nabla \times \boldsymbol{F} = \boldsymbol{0}$).*
3. *For any closed oriented simple curve Γ in Ω it holds that $\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = 0$.*

Remark. Note that (1) $\Rightarrow$ (2) follows from the observation in (5.10), while (2) $\Rightarrow$ (3) follows from Stokes' Theorem since any closed curve Γ can be viewed as the boundary of some surface Σ such that $\Gamma = \partial\Sigma$ and thus

$$\oint_\Gamma = \iint_\Sigma \underbrace{(\nabla \times \boldsymbol{F})}_{=0} \cdot d\boldsymbol{A} = 0.$$

To show that (3) $\Rightarrow$ (1), it suffices to define a scalar potential. To do this, we pick a fixed point $\boldsymbol{r}_0 \in \Omega$. For any other point $\boldsymbol{r} \in \Omega$, we may choose any oriented curve $\Gamma_{\boldsymbol{r}}$ with start point $\boldsymbol{r}_0$ and end point $\boldsymbol{r}$ and define

$$\Psi_{\boldsymbol{r}_0}(\boldsymbol{r}) = \int_{\Gamma_{\boldsymbol{r}}} \boldsymbol{F} \cdot d\boldsymbol{s}. \tag{5.11}$$

By assumption, the line integral of any closed oriented simple curve over $\boldsymbol{F}$ must be zero, and this implies that the line integrals of $\boldsymbol{F}$ are independent of path. Hence it doesn't matter which curve Γ we pick to define $\Psi_{\boldsymbol{r}_0}(\boldsymbol{r})$ in (5.11). The equation in (5.11) therefore defines a scalar field on Ω. For this scalar field, it can be shown that

$$\nabla \Psi_{\boldsymbol{r}_0}(\boldsymbol{r}) = \nabla \int_{\Gamma_{\boldsymbol{r}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \boldsymbol{F}(\boldsymbol{r}),$$

and thus $\boldsymbol{F}$ is conservative with $\Psi_{\boldsymbol{r}_0}$ as a scalar potential.