ECE 206 – University of Waterloo	Fall 2019
Lecture notes for Week 6	
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6.1 Three-dimensional integration

We have previously reviewed the main aspects of two-dimensional integration, which is the integration of a real-valued function f over a region $D \subseteq \mathbb{R}^2$ in the plane. In this section we extend these ideas to three-dimensional integration, where we will integrate real-valued functions f over regions $\Omega \subseteq \mathbb{R}^3$ in space. Three-dimensional integration is essential in many areas of physics and engineering, and will play a crucial role in Maxwell's Equations and electromagnetism.

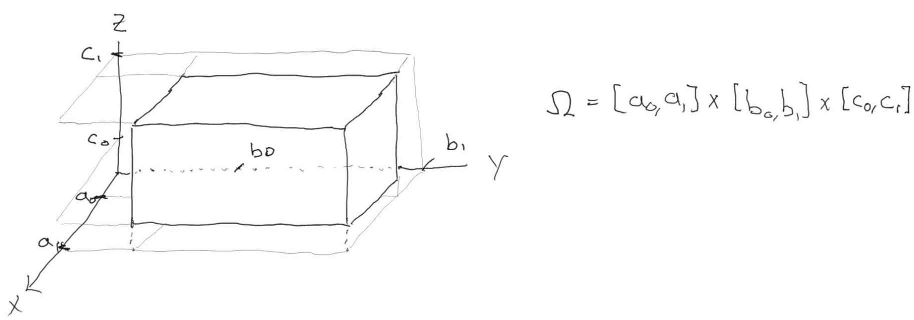

Figure 6.1: A rectangular parallelepiped in $\mathbb{R}^3$.

Suppose we have a region $\Omega \subseteq \mathbb{R}^3$ in space and a function real-valued $f : \Omega \to \mathbb{R}$. For simplicity, we will suppose that Ω is a *rectangular parallelepiped* as shown in 6.1, where the sides of the parallelepiped are the intervals

$$a_0 \leq x \leq a_1, \quad b_0 \leq y \leq b_1, \quad \text{and} \quad c_0 \leq z \leq c_1$$

for numbers $a_0 < a_1$, $b_0 < b_1$, and $c_0 < c_1$, such that the region can be denoted as

$$\Omega = [a_0, a_1] \times [b_0, b_1] \times [c_0, c_1].$$

We now define the integral of the function f over this region Ω. To this end we subdivide the region into N equally sized smaller rectangular regions $\Omega_1, \Omega_2, \ldots, \Omega_N$ each with sides of lengths Δx, Δy, and Δz, such that volume of subregion is $\text{volume}(\Omega_i) = \Delta x \Delta y \Delta z$. We can also pick a point (x_i, y_i, z_i) in each subregion. If $f(x, y, z)$ represents the *density* of matter in the space around the point (x, y, z), an approximation for the *total mass* contained in the region Ω is therefore

$$\text{total mass} \approx \sum_{i=1}^{N} f(x_i, y_i, z_i) \Delta x \Delta y \Delta z.$$

If this approximation converges to a value in the limit as $N \to 0$ and $\Delta x \Delta y, \Delta z \to 0$, the resulting limit is the integral of f over the region Ω and is denoted

$$\iiint_\Omega f \, dV \qquad \text{or} \qquad \iiint_\Omega f(x, y, z) \, dx \, dy \, dz.$$

In particular, the *volume* of a region $\Omega \subseteq \mathbb{R}^3$ can be computed as

$$\text{volume}(\Omega) = \iiint_\Omega 1\, dV$$

by taking f the be the constant function $f(x, y, z) = 1$. For regions Ω that are rectangular (as in Figure 6.1) this integral can be evaluated as a *triple integral*

$$\iiint_\Omega f\, dV = \int_{c_0}^{c_1} \left(\int_{b_0}^{b_1} \left(\int_{a_0}^{a_1} f(x, y, z)\, dx \right) dy \right) dz,$$

and (as in the case for two-dimensional integrals) it does not matter which order we integrate the variables.

6.1.1 Integration over simple regions

What if the region $\Omega \subset \mathbb{R}^3$ is not a rectangular parallelepiped? Similar to two-dimensional integration, we typically restrict our attention to *simple* regions. A z-simple region is a volume of space that is bounded between the graphs of two functions above a region in the xy-plane. An example of a z-simple region is shown in Figure 6.2

Definition 6.1. A region $\Omega \subset \mathbb{R}^3$ is *z-simple* if there is a region $D \subset \mathbb{R}^2$ in the plane and two functions $g_0 : D \to \mathbb{R}$ and $g_1 : D \to \mathbb{R}$ such that

$$\Omega = \big\{(x, y, z) \,\big|\, (x, y) \in D \text{ and } g_0(x, y) \leq z \leq g_1(x, y)\big\}.$$

Regions that are *x-simple* and *y-simple* are define analogously.

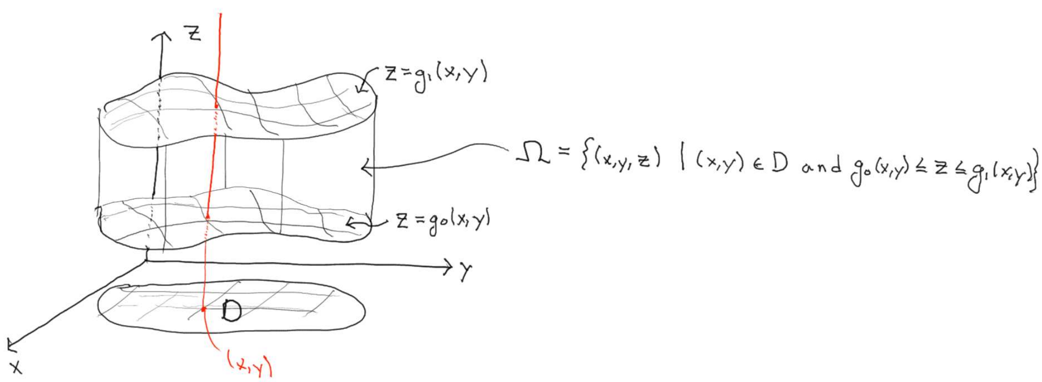

Figure 6.2: A z-simple region Ω in $\mathbb{R}^3$ that is bounded above and below by the graphs of two different functions $g_0(x, y)$ and $g_1(x, y)$ over a region D in the xy-plane.

Similar to simple regions in two dimensions, we can integrate over simple regions in three dimensions as follows. If $\Omega \subset \mathbb{R}^3$ is a z-simple region $\Omega = \{(x, y, z) \,|\, (x, y) \in D \text{ and } g_0(x, y) \leq z \leq g_1(x, y)\}$ for some functions $g_0, g_1 : D \to \mathbb{R}$ over some region $D \subseteq \mathbb{R}^2$ in the plane, we can integrate a scalar field $f : \Omega \to \mathbb{R}$ over this region by first integrating with respect to the z-variable for arbitrary fixed points $(x, y) \in D$, then integrate the resulting two-dimensional integral over D:

$$\iiint_\Omega f\, dV = \iint_D \left(\int_{g_0(x,y)}^{g_1(x,y)} f(x, y, z)\, dz \right) dx\, dy = \iint_D h(x, y)\, dx\, dy,$$

where we define the function $h : D \to \mathbb{R}$ for all $(x, y) \in D$ as

$$h(x, y) = \int_{g_0(x,y)}^{g_1(x,y)} f(x, y, z)\, dz, \tag{6.1}$$

where we perform the integration in (6.1) by holding x and y constant.

Example 6.2. Integrate $\iiint_\Omega x\, dV$ where Ω is the tetrahedron in the positive octant and inside the plane $x + y + z = 1$ as shown in Figure 6.3.

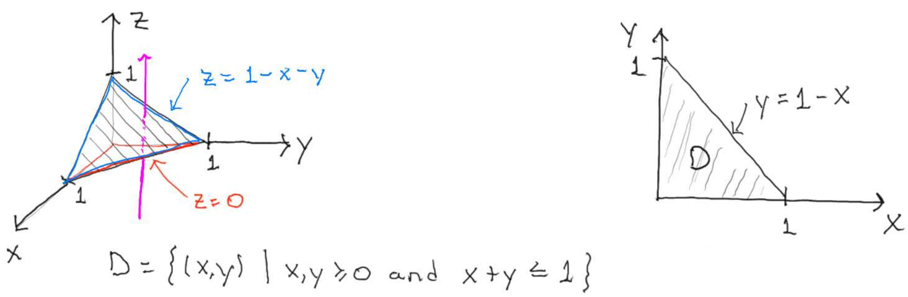

Figure 6.3: The tetrahedral region from Example 6.2, which can be viewed as a z-simple region bounded below by $g_0(x, y) = 0$ and bounded above by $g_1(x, y) = 1 - x - y$ above the region $D = \{(x, y) \,|\, x, y \geq 0 \text{ and } x + y \leq 1\}$ in the xy-plane.

Solution. We can view Ω as a z-simple region sitting above the two-dimensional region $D \subset \mathbb{R}^2$ in the xy-plane defined as

$$D = \{(x, y) \,|\, x, y \geq 0 \text{ and } x + y \leq 1\}$$

and bounded between the functions $h_0(x, y) = 0$ and $h_1(x, y) = 1 - x - y$ such that Ω can be defined as

$$\Omega = \big\{(x, y, z) \,\big|\, (x, y) \in D \text{ and } 0 \leq z \leq 1 - x - y\big\}.$$

Meanwhile, the region $D \subseteq \mathbb{R}^2$ in the xy-plane is a y-simple region between $0 \leq x \leq 1$ with $0 \leq y \leq 1 - x$. We can therefore evaluate the integral as

$$\begin{aligned}
\iiint_\Omega x\, dV &= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} x\, dz \right) dy \right) dx \\
&= \int_0^1 x \left(\int_0^{1-x} (1 - x - y)\ dy \right) dx \\
&= \int_0^1 x \left((1 - x)y - \frac{1}{2}y^2 \right) \Bigg|_0^{1-x} dx \\
&= \int_0^1 x \left((1 - x)^2 - \frac{1}{2}(1 - x)^2 \right) dx \\
&= \frac{1}{2} \int_0^1 x(1 - x)^2\, dx \\
&= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3)\, dx \\
&= \frac{1}{2} \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right) \Bigg|_0^1 = \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{24}.
\end{aligned}$$

6.1.2 Change of variables in three dimensions

Not all regions that we will want to integrate over in this course will be simple. The best method of integration is often to perform a *change of variables*, analogous to the method we introduced for two-dimensional integrals. If there is a one-to-one transformation $\mathbf{\Phi} : \mathcal{R} \to \mathbb{R}^3$ from a simple region $\mathcal{R}$ such that the three-dimensional region $\Omega \subseteq \mathbb{R}^3$ can be described as

$$\Omega = \{\mathbf{\Phi}(u, v, w) \,|\, (u, v, w) \in \mathcal{R}\},$$

then we can simplify the integral $\iiint_\Omega f\, dV$ by instead integrating over the region $\mathcal{R} = \mathbf{\Phi}^{-1}(\Omega)$ and making a change of variables.

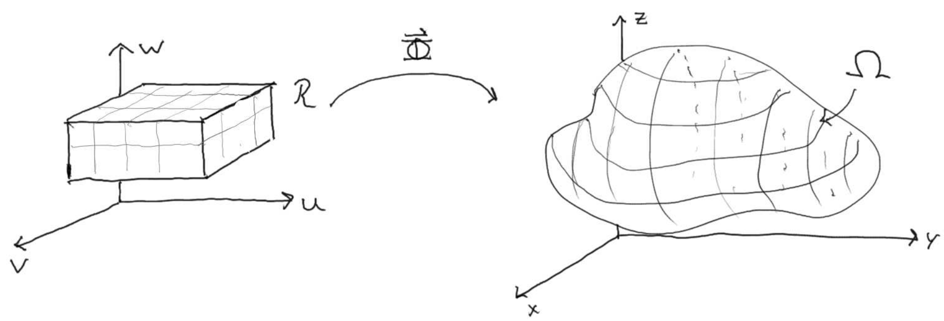

Figure 6.4: A change of variables transforming a simple region $\mathcal{R}$ into a more complicated region Ω.

We first consider a an arbitrary *parallelepiped* (as shown in Figure 6.5) in $\mathbb{R}^3$ defined by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. Its volume can be computed by

$$\text{volume} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})| = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

where we take the determinant of the the 3×3 matrix resulting from lining up the entries of the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

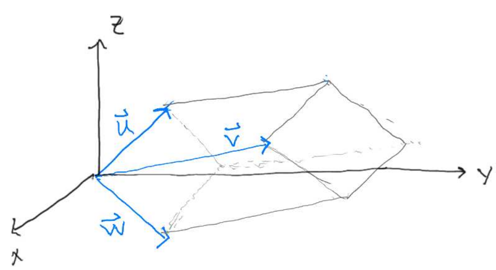

Figure 6.5: A parallelepiped in $\mathbb{R}^3$ has volume equal to $|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$.

Suppose now that some complicated region $\Omega \subseteq \mathbb{R}^3$ in xyz-space admits a parameterization $\mathbf{\Phi} : \mathcal{R} \to \mathbb{R}^3$ from some simple region $\mathcal{R}$ in uvw-space, which we may assume is a rectangular parallelepiped

$$\mathcal{R} = [a_0, a_1] \times [b_0, b_1] \times [c_0, c_1] = \{(u, v, w) \,|\, a_0 \le u \le a_1,\ b_0 \le v \le b_1,\ c_0 \le w \le c_1\}$$

for some values $a_0 < a_1$, $b_0 < b_1$, and $c_0 < c_1$. We may segment the the rectangular region $\mathcal{R}$ into many smaller equally-sized rectangular regions $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N$ each having sides with lengths Δu, Δv, and Δw.

For each $i \in \{1, \dots, N\}$, we may consider what region the small rectangular region $\mathcal{R}_i$ gets mapped to in xyz-space:

$$\Omega_i = \mathbf{\Phi}(\mathcal{R}_i).$$

We pick a point $(x_i, y_i, z_i) = \mathbf{\Phi}(u_i, v_i, w_i)$ in each small piece ω_i of the volume. For a scalar function $f : \Omega \to \mathbb{R}$, we can approximate the value of the integral

$$\iiint_\Omega f\,dV \approx \sum_{i=1}^{N} f(x_i, y_i, z_i)\ \text{volume}(\Omega_i).$$

We must now approximate the volume of each small piece Ω_i, each of which is a transformation of a small rectangular parallelepiped. For small enough values of Δu, Δv, and Δw, the small regions Ω_i are *approximately* parallelepipeds, but no longer necessarily rectangular.

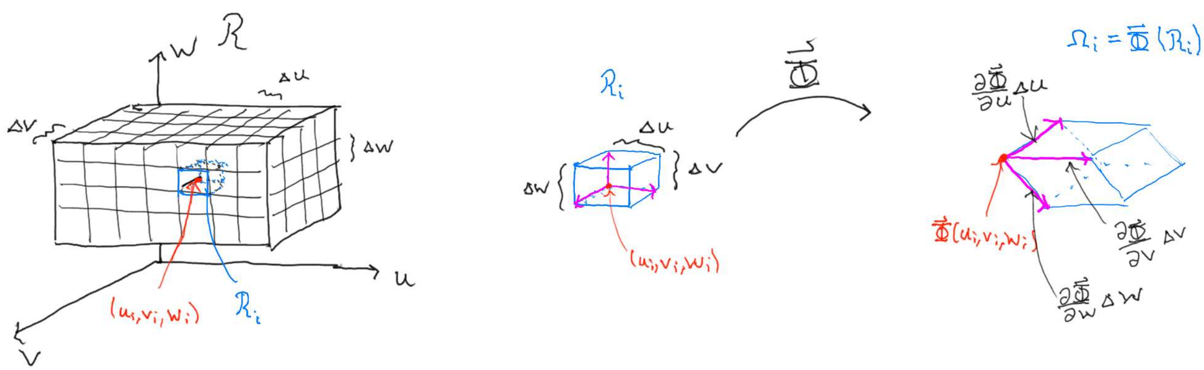

Figure 6.6: Segmenting $\mathcal{R}$ into smaller rectangular parallelepipeds $\mathcal{R}_1, \dots, \mathcal{R}_N$. Each smaller rectangular box is mapped to a small piece of the solid $\Omega_i = \mathbf{\Phi}(\mathcal{R}_i)$ which is approximately a parallelepiped with sides given by the vectors in (6.2).

If we take (u_i, v_i, w_i) to be the lower left corner of each region $\mathcal{R}_i$, we can think of $\mathcal{R}_i$ as a parallelepiped defined by the whose sides are connecting the point

$$(u_i, v_i, w_i) \quad \text{with the points} \quad (u_i + \Delta u, v_i, w_i), \quad (u_i, v_i + \Delta v, w_i), \quad \text{and} \quad (u_i, v_i, w_i + \Delta w).$$

The small piece of volume Ω_i is "approximately" a parallelepiped whose sides are given by the vectors

$$\left.\frac{\partial \mathbf{\Phi}}{\partial u}\right|_{(u_i,v_i,w_i)} \Delta u, \quad \left.\frac{\partial \mathbf{\Phi}}{\partial v}\right|_{(u_i,v_i,w_i)} \Delta v, \quad \text{and} \quad \left.\frac{\partial \mathbf{\Phi}}{\partial w}\right|_{(u_i,v_i,w_i)} \Delta w. \tag{6.2}$$

Hence the volume of Ω_i is approximately

$$\begin{aligned}
\left(\frac{\partial \mathbf{\Phi}}{\partial u}\Delta u\right) \cdot \left(\left(\frac{\partial \mathbf{\Phi}}{\partial v}\Delta v\right) \times \left(\frac{\partial \mathbf{\Phi}}{\partial w}\Delta w\right)\right) &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \cdot \left(\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \times \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right)\right) \Delta u \Delta v \Delta w \\
&= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \Delta u \Delta v \Delta w
\end{aligned}$$

where $\mathbf{\Phi} : \mathcal{R} \to \mathbb{R}^3$ has coordinates $\mathbf{\Phi}(u, v, w) = \big(x(u, v, w),\ y(u, v, w),\ z(u, v, w)\big)$. This determinant is called the *Jacobian* of the transformation.

Definition 6.3. Suppose $\mathbf{\Phi} : \mathcal{R} \to \mathbb{R}^3$ is a C^1-transformation from a region $\mathcal{R} \subseteq \mathbb{R}^3$ in space with components

$$\mathbf{\Phi}(u, v, w) = \big(x(u, v, w),\ y(u, v, w),\ z(u, v, w)\big)$$

for all $(u, v, w) \in \mathcal{R}$. The *Jacobian of the transformation* is defined as the determinant

$$J_{\mathbf{\Phi}} = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

The volume of each smaller three-dimensional region can therefore be approximated by

$$\text{volume}(\Omega_i) \approx |J_{\mathbf{\Phi}}(u_i, v_i, w_i)|\, \Delta u \Delta v \Delta w = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)}\bigg|_{(u_i, v_i, w_i)} \right| \Delta u \Delta v \Delta w$$

and the value of the desired integral can therefore be approximated by

$$\begin{aligned} \iiint_{\Omega} f\, dV &\approx \sum_{i=1}^{N} f(x_i, y_i, z_i) \text{ volume}(\Omega_i) \\ &\approx \sum_{i=1}^{N} f\big(x(u_i, v_i, w_i), y(u_i, v_i, w_i), z(u_i, v_i, w_i)\big)\, |J_{\mathbf{\Phi}}(u_i, v_i, w_i)|\, \Delta u \Delta v \Delta w. \end{aligned}$$

Taking the limit as $N \to \infty$ and $\Delta u, \Delta v, \Delta w \to 0$ gives us an integral over $\mathcal{R}$ that can be used to compute the integral by using the change of variables.

Theorem 6.4. *Suppose $\mathbf{\Phi} : \mathcal{R} \to \mathbb{R}^3$ is a one-to-one C^1-transformation from a region $\mathcal{R} \subseteq \mathbb{R}^3$ to another region $\Omega = \mathbf{\Phi}(\mathcal{R})$ and suppose $f : \Omega \to \mathbb{R}$ is a scalar field. Then*

$$\iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{R}} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw,$$

where $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is the Jacobian of the transformation as defined in Definition 6.3

6.1.3 Spherical coordinates

One of the most useful transformations of three-dimensional space, and the one that we will use most often in this course, is that of *spherical coordinates*, which is useful for parameterizing regions of space that exhibit spherical symmetry. This is an extension of our parameterization of the surface of the sphere of radius r. Each point (x, y, z) in space can be specified in spherical coordinates by its distance r from the origin

$$r = \sqrt{x^2 + y^2 + z^2},$$

then using the spherical coordinates defined earlier to denote its location on the surface of the sphere of radius r using the angles φ and θ:

$$\mathbf{\Phi}(r,\varphi,\theta) = \big(x(r,\varphi,\theta),\, y(r,\varphi,\theta),\, z(r,\varphi,\theta)\big) \qquad \text{for } 0 \le r,\ 0 \le \varphi \le \pi, \text{ and } 0 \le \theta < 2\pi$$

with components defined as

$$x(r,\varphi,\theta) = r\sin\varphi\cos\theta \qquad y(r,\varphi,\theta) = r\sin\varphi\sin\theta, \qquad \text{and} \qquad z(r,\varphi,\theta) = r\cos\varphi.$$

The Jacobian of the spherical coordinates transformation is therefore

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{vmatrix} \\ &= r^2\sin^3\varphi\cos^2\theta + r^2\sin^3\varphi\sin^2\theta z + r^2\sin\varphi\cos^2\varphi(\sin^2\theta+\cos^2\theta) \\ &= r^2\sin\varphi.\end{aligned}$$

Therefore, if a region $\Omega \subseteq \mathbb{R}^3$ in space can be parameterized using spherical coordinates over some region $\mathcal{R} = \mathbf{\Phi}^{-1}(\Omega)$, then we can compute the integral as

$$\boxed{\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_{\mathbf{\Phi}^{-1}(\Omega)} f\big(x(r,\varphi,\theta), y(r,\varphi,\theta), z(r,\varphi,\theta)\big)\, r^2\sin\varphi\,dr\,d\varphi\,d\theta}$$

where the volume element becomes $dV = r^2\sin\varphi\,dr\,d\varphi\,d\theta$.

Example 6.5. Evaluate $\displaystyle\iiint_\Omega z\,dV$ where Ω the region in the positive octant that is between the spheres of radius 1 and radius 2

$$\Omega = \left\{(x,y,z) \,\middle|\, 1 \le x^2+y^2|z^2\ 4 \text{ and } x,y,z \ge 0\right\}.$$

that is depicted in Figure 6.7.

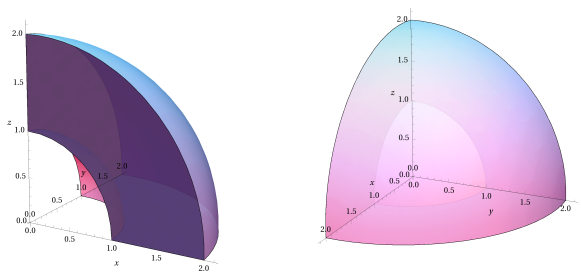

Figure 6.7: The piece of the spherical region in Example 6.5 (viewed from two different angles) that is between the spheres of radius 1 and 2 in the positive octant $x,y,z \le 0$.

Solution. We can parameterize the region Ω in spherical coordinates as the region with $1 \le r \le 2$ and angles in the intervals $0 \le \varphi \le \pi/2$ and $0 \le \theta \le \pi/2$. Since we have $z = r\cos\varphi$ in this parameterization, the integral becomes

$$\begin{aligned}\iiint_\Omega z\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (r\cos\varphi)(r^2\sin\varphi)\,dr\,d\varphi\,d\theta \\ &= \left(\int_1^2 r^3\,dr\right)\left(\int_0^{\pi/2}\cos\varphi\sin\varphi\right)\int_0^{\pi/2} d\theta \\ &= \left(\frac{1}{2}r^4\Big|_1^2\right)\left(\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/2}\right)\left(\theta\Big|_0^{\pi/2}\right) = \frac{16-1}{4}\frac{1}{2}\frac{\pi}{2} = \frac{15\pi}{16}.\end{aligned}$$

Example 6.6. Evaluate $\iiint_\Omega (x^2+y^2+z^2)^{3/2}\,dV$ where Ω the "ice cream cone"-shaped region that is inside the hemisphere defined by $z=\sqrt{8-x^2-y^2}$ and above the cone defined by $z=\sqrt{x^2+y^2}$,

$$\Omega = \left\{(x,y,z)\,\middle|\,\sqrt{x^2+y^2}\le z\le\sqrt{8-x^2-y^2}\right\},$$

that is depicted in Figure 6.8.

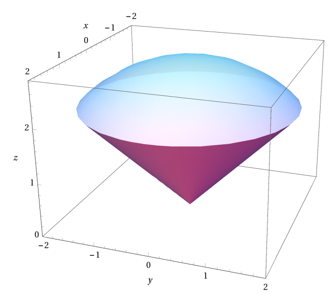

Figure 6.8: The "ice cream cone"-shaped spherical region spherical region in Example 6.6 that is inside the sphere of radius $2\sqrt{2}$ and above the cone defined by $z=\sqrt{x^2+y^2}$.

Solution. We see that the points on the surface of the cone defining the lower boundary to this region make an angle of $\pi/4$ with the z-axis, so we can describe this region in spherical coordinates where the φ-variable is in the interval $0\le\varphi\le\pi/4$, the other angle goes around the circle $0\le\theta\le 2\pi$, and the radius varies from $0\le r\le 2\sqrt{2}$ (since the sphere in the question is described by $x^2+y^2+z^2=8$). Since

$$(x^2+y^2+z^2)^{3/2} = r^3,$$

we can write the integral as

$$\begin{aligned}\iiint_\Omega (x^2+y^2+z^2)^{3/2}\,dx\,dy\,dz &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} (r^3)(r^2\sin\varphi)\,dr\,d\varphi\,d\theta \\ &= \left(\int_0^{2\sqrt{2}} r^5 dr\right)\left(\int_0^{\pi/4}\sin\varphi\,d\varphi\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \left(\frac{1}{6}r^6\Big|_0^{2\sqrt{2}}\right)\left(-\cos\varphi\Big|_0^{\pi/4}\right)\left(\theta\Big|_0^{2\pi}\right) \\ &= \left(\frac{2^9}{6}\right)\left(1-\frac{1}{\sqrt{2}}\right)2\pi \\ &= \frac{256}{3}\left(2-\sqrt{2}\right)\pi.\end{aligned}$$

6.1.4 Cylindrical coordinates

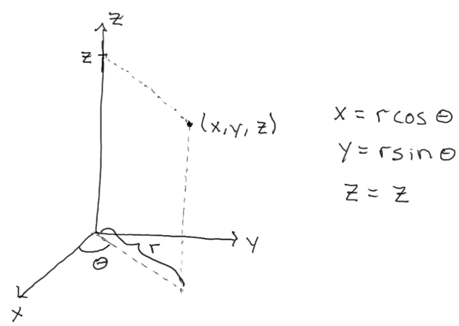

Figure 6.9: Cylindrical coordinates specify each point in $\mathbb{R}^3$ using the parameters r, θ, and z.

Another common parameterization of space makes use of *cylindrical coordinates*. In this parameterization, each point in space is specified in terms of its "height" above the xy-plane and uses polar coordinates to indicate its position on the xy-plane. This parameterization is given by

$$\boldsymbol{\Phi}(r,\theta,z) = \big(x(r,\theta),\ y(r,\theta),\ z\big) \qquad \text{for } 0 \le r,\ 0 \le \theta \le 2\pi,\ \text{ and } z \in \mathbb{R}$$

with the x- and y-components defined as

$$x(r,\theta) = r\cos\theta \qquad y(r,\theta) = r\sin\theta,$$

where the parameters are r, θ, and z. The Jacobian of this transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, if a region $\Omega \subseteq \mathbb{R}^3$ in space can be parameterized using cylindrical coordinates over some region $\mathcal{R} = \boldsymbol{\Phi}^{-1}(\Omega)$, then we can compute the integral as

$$\boxed{\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_{\boldsymbol{\Phi}^{-1}(\Omega)} f\big(x(r,\theta), y(r,\theta), z\big)\,r\,dr\,d\theta\,dz}$$

where the volume element becomes $dV = r\,dr\,d\theta\,dz$.

Example 6.7. Integrate $\iiint_\Omega (x^2+y^2)\,dV$ where Ω is the conical region

$$\Omega = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le 2 \text{ and } x, y \ge 0\}$$

depicted in Figure 6.10.

Solution. We can parameterize this region using cylindrical coordinates as the region inside the cylinder of radius 2 with z-coordinate bounded by $r \le z \le 2$ (where θ varies from 0 to 2π). Alternatively, for a fixed z,

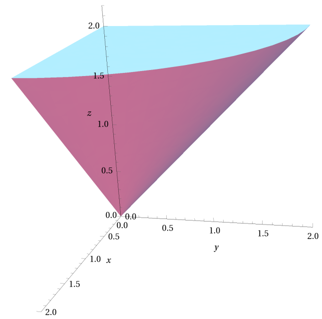

Figure 6.10: The conical region in Example 6.7.

this is the region where the radius varies from 0 to z. Since $4 = \sqrt{x^2+y^2}$ in this parameterization, we can evaluate this integral as

$$\begin{aligned}\iiint_\Omega (x^2+y^2)\,dx\,dy\,dz &= \iiint_{\mathbf{\Phi}^{-1}(\Omega)} (r^2)(r)\,dr\,d\theta\,dz \\ &= \int_0^2 \left(\int_0^{2\pi}\left(\int_0^z r^3\,dr\right)d\theta\right)dz \\ &= \left(\int_0^2 \frac{1}{4}z^4\,dz\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \left(\frac{1}{20}z^5\Big|_0^2\right)(2\pi) = \frac{16\pi}{5}\end{aligned}$$

6.2 Divergence and Gauss' Theorem

We have so far introduced a few different notions of 'derivatives' of multivariate functions in space. The *gradient* of a scalar field is a vector field that points in the direction of greatest increase at each point, while the *curl* of a vector field points in the direction greatest angular velocity that results from the spinning of the vector field at that point. The final way that we can use the del operator to define a notion of a derivative is the *divergence*.

6.2.1 Divergence

Definition 6.8. Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a C^1-vector field on a region $\Omega \subseteq \mathbb{R}^3$ with components

$$\boldsymbol{F}(x,y,z) = \big(F_1(x,y,z),\, F_2(x,y,z),\, F_3(x,y,z)\big).$$

The *divergence* of $\boldsymbol{F}$ is the scalar field $\operatorname{div}\boldsymbol{F}$ defined at each point $(x,y,z) \in \Omega$ as

$$(\operatorname{div}\boldsymbol{F})(x,y,z) = \frac{\partial F_1}{\partial x}\bigg|_{(x,y,z)} + \frac{\partial F_2}{\partial y}\bigg|_{(x,y,z)} + \frac{\partial F_3}{\partial z}\bigg|_{(x,y,z)}. \qquad (6.3)$$

Just as the curl can be computed as the "cross product" of the del operator ∇ with the vector field, we can compute the divergence of a vector field as the "dot product" of the del operator with the vector field

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where the vector field has components defined by $\boldsymbol{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$.

Remark 6.9. What is the physical significance of the divergence $\nabla \cdot \boldsymbol{F}$ of a vector field $\boldsymbol{F}$? This is by no means immediately obvious from the defining formula (6.3). We shall see from the Divergence Theorem of Gauss-Ostogradskii, to be established later in this section, that the scalar value $\nabla \boldsymbol{F}(x, y, z)$ at a point (x, y, z) in $\mathbb{R}^3$ effectively measures the local "divergence" (or "flowing away") of the vector field $\boldsymbol{F}$ out from the point (x, y, z). This local divergence is an important property for electric and magnetic fields, as we shall see later.

6.2.2 Boundaries of regions and closed surfaces

Before we discuss the physical significance of the Divergence Theorem (and how to derive it) in the next subsection, we must first introduce the concept of a *boundary of a three-dimensional region*. Given a three-dimensional region $\Omega \subset \mathbb{R}^3$ in space, its *boundary* is an oriented two-dimensional surface denoted by

$$\text{boundary of } \Omega = \partial\Omega.$$

The orientation of the surface $\partial\Omega$ is always taken to be "outward" and points into space away from the interior of the region.

Example 6.10. The solid region that is the ball of radius r centered at the origin

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq r^2\}$$

has as its boundary surface

$$\partial\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = r^2\},$$

which is the surface of the sphere of radius r centered at the origin.

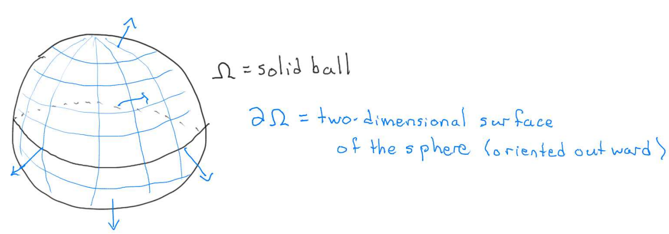

Figure 6.11: The boundary of a three-dimensional solid is a two-dimensional surface with outward-facing orientation. The boundary of a solid ball is the sphere of the same radius.

Definition 6.11. A surface $\Sigma \subseteq \mathbb{R}^3$ is *closed* if it is the boundary of some solid region $\Omega \subseteq \mathbb{R}^3$ such that $\Sigma = \partial\Omega$.

Similar to line integrals of vector fields over closed curves, we have a special notation for the flux integral of a vector field over closed surface. If the surface Σ is closed, the *total flux* of a vector field $\boldsymbol{F}$ through Σ is denoted

$$\oiint_{\Sigma} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA.$$

More complicated regions will have more complicated boundaries, and it is common to describe the different pieces of the boundary of a region. For example, consider region that is the cylindrical solid

$$\Omega = \{(x, y, z) \,|\, x^2 + y^2 \leq r \text{ and } 0 \leq z \leq h\},$$

which is the region inside a cylinder of radius r along the z-axis and has height h. The boundary of this solid has three pieces: the cylindrical side and the top and bottom discs, which we can label (as in Figure 6.12) as the side

$$\Sigma_1 = \{(x, y, z) \,|\, x^2 + y^2 = r^2 \text{ and } 0 \leq z \leq h\}$$

and the top and bottom

$$\Sigma_2 = \{(x, y, z) \,|\, x^2 + y^2 \leq r^2 \text{ and } z = 0\} \qquad \Sigma_3 = \{(x, y, z) \,|\, x^2 + y^2 \leq r^2 \text{ and } z = h\}$$

such that the entire boundary of Ω is described as

$$\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

An integral of a vector field $\boldsymbol{F}$ over the boundary of Ω can therefore be computed by summing up the integrals over each part as

$$\oiint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{A} = \iint_{\Sigma_1} \boldsymbol{F} \cdot d\boldsymbol{A} + \iint_{\Sigma_2} \boldsymbol{F} \cdot d\boldsymbol{A} + \iint_{\Sigma_3} \boldsymbol{F} \cdot d\boldsymbol{A},$$

where the orientation of each piece points outward from the interior of the cylinder.

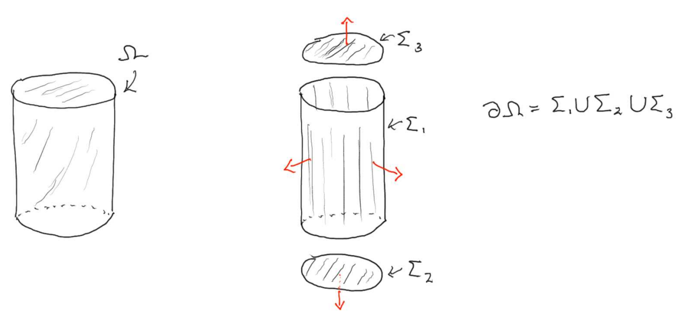

Figure 6.12: The boundary of the cylinder has three pieces: the side, the top, and the bottom.

6.2.3 Gauss-Ostogradskii Theorem

We now come to the second major theorem of vector calculus: the *Theorem of Gauss-Ostrogradskii* or the *Divergence Theorem*. In contrast to Stokes' theorem, which involves the surface integral of the curl of a vector field over a surface, the divergence theorem involves the surface integral of a vector field over a closed surface. After proving the Divergence Theorem for rectangular regions, we will use it to help us understand the physical significance of the divergence in the following subsection.

Theorem 6.12 (Divergence Theorem of Gauss-Ostogradskii)**.** *Suppose $\Omega \subseteq \mathbb{R}^3$ is a region in space and that $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is a C^1-vector field in Ω. It holds that*

$$\iiint_{\Omega} \nabla \cdot \boldsymbol{F}\, dV = \oiint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{A}.$$

We now present a proof of the Divergence Theorem for rectangular regions. Suppose $\Omega \subseteq \mathbb{R}^3$ is a rectangular region

$$\Omega = [a_0, a_1] \times [b_0, b_1] \times [c_0, c_1] = \{(x, y, z) \,|\, a_0 \leq x \leq a_1,\ b_0 \leq y \leq b_1,\ c_0 \leq z \leq c_1\}$$

for some values $a_0 < a_1$, $b_0 < b_1$, and $c_0 < c_1$. Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a C^1-vector field in this region. The boundary of this rectangular cube region has six parts: top, bottom, front, back, left, and right. We can label these surfaces $\Sigma_{\text{top}}, \Sigma_{\text{bottom}} \ldots, \Sigma_{\text{front}}$ as $\Sigma_1, \Sigma_2, \ldots, \Sigma_6$ (as in Figure 6.13) where

$$\begin{aligned}
\Sigma_{\text{top}} &= \Sigma_1 = \{(x, y, c_1) \,|\, a_0 \leq x \leq a_1 \text{ and } b_0 \leq y \leq b_1\} \\
\Sigma_{\text{bottom}} &= \Sigma_2 = \{(x, y, c_0) \,|\, a_0 \leq x \leq a_1 \text{ and } b_0 \leq y \leq b_1\} \\
\Sigma_{\text{right}} &= \Sigma_3 = \{(x, b_0, z) \,|\, a_0 \leq x \leq a_1 \text{ and } c_0 \leq z \leq c_1\} \\
\Sigma_{\text{left}} &= \Sigma_4 = \{(x, b_1, z) \,|\, a_0 \leq x \leq a_1 \text{ and } c_0 \leq z \leq c_1\} \\
\Sigma_{\text{back}} &= \Sigma_5 = \{(a_0, y, z) \,|\, b_0 \leq y \leq b_1 \text{ and } c_0 \leq z \leq c_1\} \\
\Sigma_{\text{front}} &= \Sigma_6 = \{(a_1, y, z) \,|\, b_0 \leq y \leq b_1 \text{ and } c_0 \leq z \leq c_1\}
\end{aligned}$$

such that the entire boundary of the rectangular region is given by

$$\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6.$$

Note that each part of the surface has *outward* orientation. In particular,

- the top Σ_{top} has normal vector $\hat{\boldsymbol{n}}_{\text{top}} = \hat{\boldsymbol{k}}$ and the bottom Σ_{bot} has normal vector $\hat{\boldsymbol{n}}_{\text{bot}} = -\hat{\boldsymbol{k}}$,
- the right face Σ_{right} has normal vector $\hat{\boldsymbol{n}}_{\text{right}} = \hat{\boldsymbol{\jmath}}$ and the left face Σ_{left} has normal vector $\hat{\boldsymbol{n}}_{\text{left}} = -\hat{\boldsymbol{\jmath}}$,
- and the front Σ_{front} has normal vector $\hat{\boldsymbol{n}}_{\text{front}} = \hat{\boldsymbol{\imath}}$ and the back Σ_{back} has normal vector $\hat{\boldsymbol{n}}_{\text{back}} = -\hat{\boldsymbol{\imath}}$.

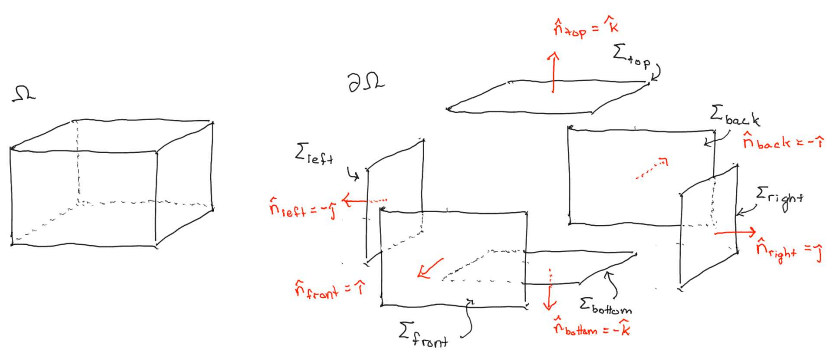

Figure 6.13: The rectangular region Ω has six parts to its boundary.

We can now compute the flux integral through the entire surface of the boundary by summing the fluxes from each individual piece,

$$\oiint_{\partial\Omega} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA = \iint_{\Sigma_{\text{top}}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}_{\text{top}}\, dA + \cdots + \iint_{\Sigma_{\text{front}}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}_{\text{front}}\, dA.$$

If we define the components of the vector field as $\boldsymbol{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$, note that

$$\boldsymbol{F}(x,y,z)\cdot\hat{\boldsymbol{\imath}} = F_1(x,y,z), \qquad \boldsymbol{F}(x,y,z)\cdot\hat{\boldsymbol{\jmath}} = F_2(x,y,z), \qquad \text{and} \qquad \boldsymbol{F}(x,y,z)\cdot\hat{\boldsymbol{k}} = F_3(x,y,z).$$

We now separate the integration into the components.

- On points on the top and bottom of the cube, the field takes on the values $\boldsymbol{F}(x,y,c_1)$ and $\boldsymbol{F}(x,y,c_0)$. The flux through the top and bottom is therefore computed as

$$\begin{aligned}\iint_{\Sigma_{\text{top}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{top}}\,dA + \iint_{\Sigma_{\text{bot}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{bot}}\,dA &= \iint_{\Sigma_{\text{top}}} \boldsymbol{F}\cdot\hat{\boldsymbol{k}}\,dA + \iint_{\Sigma_{\text{bot}}} \boldsymbol{F}\cdot(-\hat{\boldsymbol{k}})\,dA \\ &= \int_{b_0}^{b_1}\int_{a_0}^{a_1} \left(F_3(x,y,c_1) - F_3(x,y,c_0)\right)\,dx\,dy \\ &= \int_{b_0}^{b_1}\int_{a_0}^{a_1} \left(F_3(x,y,z)\Big|_{z=c_0}^{c_1} \right)\,dx\,dy \\ &= \int_{b_0}^{b_1}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \left(\frac{\partial F_3}{\partial z}\Big|_{(x,y,z)} \right)\,dz\,dx\,dy \\ &= \iiint_{\Omega} \frac{\partial F_3}{\partial z}\,dV,\end{aligned}$$

  where in the second-to-last line we use the fact that $\boldsymbol{F}$ is C^1 and the Fundamental Theorem of Calculus to see that

$$\int_{c_0}^{c_1} \left(\frac{\partial F_3}{\partial z}\Big|_{(x,y,z)} \right)\,dz = F_3(x,y,z)\Big|_{z=c_0}^{c_1}.$$

- On points on the right and left faces of the cube, the field takes on the values $\boldsymbol{F}(x,b_1,z)$ and $\boldsymbol{F}(x,b_1,z)$. The flux through the right and left faces is therefore computed as

$$\begin{aligned}\iint_{\Sigma_{\text{right}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{right}}\,dA + \iint_{\Sigma_{\text{left}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{left}}\,dA &= \iint_{\Sigma_{\text{right}}} \boldsymbol{F}\cdot\hat{\boldsymbol{\jmath}}\,dA + \iint_{\Sigma_{\text{left}}} \boldsymbol{F}\cdot(-\hat{\boldsymbol{\jmath}})\,dA \\ &= \int_{c_0}^{c_1}\int_{a_0}^{a_1} \left(F_2(x,b_1,z) - F_2(x,b_0,z)\right)\,dx\,dz \\ &= \int_{c_0}^{c_1}\int_{a_0}^{a_1}\int_{b_0}^{b_1} \left(\frac{\partial F_2}{\partial y}\Big|_{(x,y,z)} \right)\,dy\,dx\,dy \\ &= \iiint_{\Omega} \frac{\partial F_2}{\partial y}\,dV,\end{aligned}$$

  using the same strategy as for the top and bottom parts.

- Finally, on points on the front and back faces of the cube, the field takes on the values $\boldsymbol{F}(a_1,y,z)$ and $\boldsymbol{F}(a_0,y,z)$. The flux through the right and left faces is therefore computed as

$$\begin{aligned}\iint_{\Sigma_{\text{front}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{front}}\,dA + \iint_{\Sigma_{\text{back}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{back}}\,dA &= \iint_{\Sigma_{\text{front}}} \boldsymbol{F}\cdot\hat{\boldsymbol{\imath}}\,dA + \iint_{\Sigma_{\text{back}}} \boldsymbol{F}\cdot(-\hat{\boldsymbol{\imath}})\,dA \\ &= \int_{c_0}^{c_1}\int_{b_0}^{b_1} \left(F_1(a_1,y,z) - F_1(a_1,y,z)\right)\,dy\,dz \\ &= \int_{c_0}^{c_1}\int_{b_0}^{b_1}\int_{a_0}^{a_1} \left(\frac{\partial F_1}{\partial x}\Big|_{(x,y,z)} \right)\,dx\,dy\,dy \\ &= \iiint_{\Omega} \frac{\partial F_1}{\partial x}\,dV.\end{aligned}$$

Summing up the fluxes over all of the pieces of the boundary, we find that

$$\begin{aligned}\oiint_{\partial\Omega} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dA &= \iint_{\Sigma_{\text{front}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{front}}\,dA + \iint_{\Sigma_{\text{back}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{back}}\,dA + \iint_{\Sigma_{\text{right}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{right}}\,dA + \iint_{\Sigma_{\text{left}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{left}}\,dA \\ &\quad + \iint_{\Sigma_{\text{top}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{top}}\,dA + \iint_{\Sigma_{\text{bot}}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}_{\text{bot}}\,dA \\ &= \iiint_{\Omega} \frac{\partial F_1}{\partial x}\,dV + \iiint_{\Omega} \frac{\partial F_2}{\partial y}\,dV + \iiint_{\Omega} \frac{\partial F_3}{\partial z}\,dV \\ &= \iiint_{\Omega} \nabla\cdot\boldsymbol{F}\,dV,\end{aligned}$$

as desired.

Remark 6.13. Previously we saw how Green's Theorem could be used to give physical significance to the 'vorticity' of a vector field in $\mathbb{R}^2$ (which gives us an analogous interpretation of the curl in $\mathbb{R}^3$). We can now use the Gauss-Ostogradskii Theorem to give us a physical meaning of the divergence. Consider a C^1-vector field $\boldsymbol{F}$ and a point $\boldsymbol{r}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ in space. If we take a very small spherical region Ω_ε of radius ε centered at $\boldsymbol{r}_0$ and compute the flux of the vector field outward through the surface of this region, we use the Divergence Theorem to obtain

$$\oiint_{\partial\Omega_\varepsilon} \boldsymbol{F}\cdot d\boldsymbol{A} = \iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{F}\,dV.$$

If we take ε to be very small, the value on the right-hand side of the above equation can be approximated by

$$\iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{F}\,dV \approx \nabla\cdot\boldsymbol{F}(x_0,y_0,z_0)\ \text{volume}(\Omega_\varepsilon)$$

since the divergence of $\boldsymbol{F}$ is approximately constant over a small enough region Ω_ε. Rearranging the terms, we get that

$$\boldsymbol{F}(x_0,y_0,z_0) \approx \frac{1}{\text{volume}(\Omega_\varepsilon)} \iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{F}\,dV,$$

and taking the limit as $\varepsilon \to 0$ gives us an alternative definition for the divergence:

$$\nabla\cdot\boldsymbol{F}(x_0,y_0,z_0) = \lim_{\varepsilon\to 0} \frac{1}{\text{volume}(\Omega_\varepsilon)} \iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{F}\,dV.$$

This effectively says that the divergence $\nabla\cdot\boldsymbol{F}(x,y,z)$ is approximately the flux of $\boldsymbol{F}$ through the small spherical surface $\Sigma_\varepsilon = \partial\Omega_\varepsilon$ centered at (x,y,z) per volume of the region Ω_ε enclosed by Σ_ε.

- If $\nabla\cdot\boldsymbol{F}(x,y,z) > 0$, then the point (x,y,z) is called a *source*.
- If $\nabla\cdot\boldsymbol{F}(x,y,z) < 0$, then the point (x,y,z) is called a *sink*.

This gives us a physical interpretation of the Divergence Theorem. "The total outward flow of a vector field through the boundary of a three-dimensional region is equal to the sum of the microscopic sources of the flow at each point inside the region." See Figure 6.14 for depictions of examples of a source and a sink.

Remark 6.14. Suppose that a vector field $\boldsymbol{J}$ represents the *current density* (in units of coulomb per cubed meter per second $= \mathrm{C}\cdot\mathrm{m}^{-3}\cdot\mathrm{s}^{-1}$ or amps per curbed meter $= \mathrm{A}\cdot\mathrm{m}^{-3}$) of points in space. (Essentially, the density and direction of current flowing in space.) Then the flux integral

$$\text{current through } \Sigma = \iint_{\Sigma} \boldsymbol{J}\cdot d\boldsymbol{A}$$

represents the *total current* (in units of amps) flowing through the surface Σ. If $\nabla\cdot\boldsymbol{J} > 0$ at a point, then we must have charge flowing outward and away from that point, while $\nabla\cdot\boldsymbol{J} < 0$ indicates that charge is flowing toward that point.

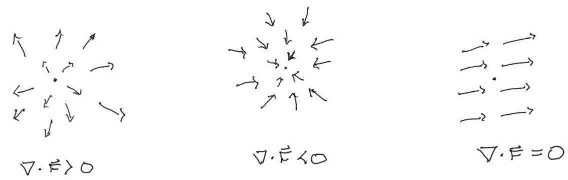

Figure 6.14: A *source* of a vector field is a point with positive divergence while a *sink* is a point with negative divergence.

Example 6.15. Suppose the current density in a region of space is given by $\boldsymbol{J}(x,y,z) = (x^3, -2xy, yz)$. Let Ω_ε be a small spherical region of radius ε equal to

$$\varepsilon = 10^{-6}\text{m}$$

centered at the point $(x,y,z) = (1,2,1)$ (with all distances given in meters). What is the total current flowing outward through the boundary of Ω_ε? We *could* parameterize the surface of the sphere and compute the resulting flux integral, but it is much easier to use the Divergence Theorem and approximate the resulting volume integral

$$\oiint_{\partial\Omega_\varepsilon} \boldsymbol{J}\cdot d\boldsymbol{A} = \iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{J}\,dV.$$

Note that the divergence of $\boldsymbol{J}$ is

$$\begin{aligned}\nabla\cdot\boldsymbol{J} &= \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}\\ &= 3x^2 - 2x + y,\end{aligned}$$

and thus the divergence is $\nabla\cdot\boldsymbol{J}(1,2,1) = 3$ at this point. Since ε is very small, we can approximate the value of this volume integral as

$$\begin{aligned}\text{total current} = \iiint_{\Omega_\varepsilon} \nabla\cdot\boldsymbol{J}\,dV &\approx \nabla\cdot\boldsymbol{J}(1,2,1)\ \text{volume}(\Omega_\varepsilon)\\ &= 3\left(\frac{4}{3}\pi\varepsilon^3\right) = 4\pi 10^{-18}\ \text{amps}.\end{aligned}$$

6.3 Solenoidal vector fields and vector potential

Recall that a vector field $\boldsymbol{F}$ is *conservative* if there exists a scalar field Ψ such that $\boldsymbol{F} = \nabla\Psi$. Moreover, note that the curl of any gradient field is always equal to zero:

$$\nabla\times(\nabla\Psi) = \mathbf{0}.$$

Hence any conservative field is *curl-free* (or *irrotational*). In fact (as we have seen in last week's notes), for a vector field $\boldsymbol{F}:\mathbb{R}^3\to\mathbb{R}^3$,

$$\nabla\times\boldsymbol{F} = \mathbf{0}\ \text{everywhere} \qquad\Longleftrightarrow\qquad \boldsymbol{F}\ \text{is conservative (i.e., } \boldsymbol{F} = \nabla\Psi).$$

A scalar field Ψ such that $\boldsymbol{F} = \nabla\Psi$ is called a *scalar potential* field for $\boldsymbol{F}$. Scalar potentials are not unique. Indeed, if $\Psi(x,y,z)$ is a scalar potential for $\boldsymbol{F}(x,y,z)$, then so is $\Psi(x,y,z)+c$ for any constant c. Something similar can be said about *divergence-free* fields.

> **Definition 6.16.** A C^1-vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ on a region Ω is said to be *solenoidal* (or *divergence-free*) if $\nabla \cdot \boldsymbol{F} = 0$ everywhere.

Remark 6.17. Let $\boldsymbol{F}$ be a vector field and suppose there is another C^2-vector field $\boldsymbol{G}$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$ which has components $\boldsymbol{G}(x,y,z) = (G_1(x,y,z), G_2(x,y,z), G_3(x,y,z))$. Then the divergence of $\boldsymbol{F}$ is

$$\begin{aligned}
\nabla \cdot \boldsymbol{F} &= \nabla \cdot (\nabla \times \boldsymbol{G}) \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\hat{\boldsymbol{k}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) \\
&= \underbrace{\frac{\partial^2 G_1}{\partial y \partial z} - \frac{\partial^2 G_1}{\partial z \partial y}}_{=0} + \underbrace{\frac{\partial^2 G_2}{\partial z \partial x} - \frac{\partial^2 G_2}{\partial x \partial z}}_{=0} + \underbrace{\frac{\partial^2 G_3}{\partial x \partial y} - \frac{\partial^2 G_3}{\partial y \partial x}}_{=0} \\
&= 0,
\end{aligned}$$

where we make use of the fact that G_1, G_2, and G_3 have continuous second order partial derivatives (since $\boldsymbol{G}$ was assumed to be C^2) and thus the order of differentiation does not matter. It follows that the curl of any vector field (i.e., any vector field of the form $\boldsymbol{F} = \nabla \times \boldsymbol{G}$) is solenoidal. For C^1-vector fields defined on all of $\mathbb{R}^3$, it turns out the the reverse is true as well! We state this fact here, but we will not prove it.

> **Theorem 6.18.** *Suppose $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field. The following are equivalent:*
>
> 1. $\nabla \cdot \boldsymbol{F} = \boldsymbol{0}$ *everywhere.*
> 2. *There exists a C^2-vector field $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$ everywhere.*

A vector field $\boldsymbol{G}$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$ is called a *vector potential* field for $\boldsymbol{F}$. Just as scalar potentials are not unique, neither are vector potentials. Indeed, if $\boldsymbol{G}$ is a vector potential field for $\boldsymbol{F}$, then so is $\boldsymbol{G} + \nabla f$ for any C^1-function $f : \mathbb{R}^3 \to \mathbb{R}$, since

$$\nabla \times (\boldsymbol{G} + \nabla f) = \underbrace{\nabla \times \boldsymbol{G}}_{=\boldsymbol{F}} + \underbrace{\nabla \times (\nabla f)}_{=\boldsymbol{0}} = \boldsymbol{F}.$$

Thus there are infinitely many choices for the vector potential $\boldsymbol{G}$ for a given solenoidal vector field $\boldsymbol{F}$. We often choose the one that is most convenient for calculations.

6.3.1 Finding a vector potential

Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a solenoidal C^1-vector field. A vector potential for $\boldsymbol{F}$ can be computed using the following methodology. Write the components of $\boldsymbol{F}$ as $\boldsymbol{F} = (F_1, F_2, F_3)$, and suppose $\boldsymbol{G} = (G_1, G_2, G_3)$ is a vector potential field for $\boldsymbol{F}$. Solving $\nabla \times \boldsymbol{G} = \boldsymbol{F}$ for $\boldsymbol{G}$ amounts to simultaneously solving the following system of partial differential equations:

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = F_1$$
$$\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = F_2$$
$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3$$

To solve this, we use the fact that $\boldsymbol{A}$ is not unique. Suppose $\boldsymbol{H} = (H_1, H_2, H_3)$ is also a vector potential for $\boldsymbol{F}$, then so is $\boldsymbol{H} + \nabla f$ for any C^1 function f. We may choose f so that $\frac{\partial f}{\partial z} = -H_3$, and thus

$$\boldsymbol{H} + \nabla f = \left(H_1 - \frac{\partial f}{\partial x},\, H_2 - \frac{\partial f}{\partial y},\, 0\right)$$

is a vector potential field for $\boldsymbol{F}$. We may therefore assume that $\boldsymbol{G}$ has the simple form $\boldsymbol{G} = (G_1, G_2, 0)$, which greatly simplifies the computation! Thus we only need to solve the following system of partial differential equations:

$$-\frac{\partial G_2}{\partial z} = F_1$$
$$\frac{\partial G_1}{\partial z} - = F_2$$
$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3.$$

We illustrate how to do this with the following example.

Example 6.19. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by $\boldsymbol{F}(x, y, z) = (x^2, 3xz, -2xz)$. Show that $\boldsymbol{F}$ is solenoidal and find a vector potential for $\boldsymbol{F}$.

Solution. Firstly, we note that $\nabla \cdot \boldsymbol{F} = 2x + 0 - 2x = 0$, so $\boldsymbol{F}$ is indeed solenoidal. Next, assume that $\boldsymbol{G} = (G_1, G_2, 0)$ is a vector potential. Then

$$-\frac{\partial G_2}{\partial z} = x^2 \qquad \Longrightarrow \qquad G_2(x, y, z) = x^2 z + g(x, y)$$
$$\frac{\partial G_1}{\partial z} = 3xz^2 \qquad \Longrightarrow \qquad G_1(x, y, z) = 3xz^2 + h(x, y)$$

where g and h are some differentiable functions that depend only on x and y. Moreover, $\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3$ implies that

$$\left(-2xz + \frac{\partial g}{\partial x}\right) - \left(0 + \frac{\partial h}{\partial y}\right) = -2xz$$

or $\frac{\partial df}{\partial x} = \frac{\partial h}{\partial y}$, so it remains to pick functions $g(x, y)$ and $h(x, y)$ that satisfy this condition. We can simply pick $g(x, y) = h(x, y) = 0$. There are infinitely many choices that will do here, but this is certainly the simplest! Hence, the vector field $\boldsymbol{G}(x, y, z) = (3xz^2, x^2 z, 0)$ is a vector potential for $\boldsymbol{F}$. You should compute $\nabla \times \boldsymbol{G}$ to verify that this is actually equal to $\boldsymbol{F}$.

Remark 6.20. As we will learn later, the magnetic field $\boldsymbol{B}$ in the study of electromagnetism is always solenoidal. A vector field $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

is called a *magnetic potential field* for $\boldsymbol{B}$. However, one typically takes a different approach to finding these. A magnetic field usually results from a current (a stream of moving charged particles). This fact can be used to find a vector potential that is more physically natural.

6.4 Vector calculus identities

Here we collect some useful identities in vector calculus. These formulas will be included on the exam formula sheets, so you don't need to memorize them, but make sure you know how to prove them! In the following, let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.

6.4.1 Product rules for vector derivatives

Here we have two useful product rules for vector derivatives. Proofs of these will be found in the practice problems.

$$\nabla \times (f\boldsymbol{F}) = (\nabla f) \times \boldsymbol{F} + f(\nabla \times \boldsymbol{F}) \tag{6.4}$$

$$\nabla \cdot (f\boldsymbol{F}) = (\nabla f) \cdot \boldsymbol{F} + f(\nabla \cdot \boldsymbol{F}) \tag{6.5}$$

6.4.2 Second derivative identities for vector derivatives

For a scalar field f, note that the gradient ∇f is itself a vector field. Since the gradient is a vector field, and we have two different ways of taking the derivative of a vector field (curl and divergence), we can find *second derivative* rules for the del operator ∇. We already know that the curl of a gradient field is zero:

$$\nabla \times (\nabla f) = \mathbf{0}. \tag{6.6}$$

However, we can also take the divergence of a gradient field. Expanding it out, we get

$$\begin{aligned} \nabla \cdot (\nabla f) &= \nabla \cdot \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \nabla^2 f, \end{aligned}$$

where ∇^2 is the *Laplacian operator* defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{6.7}$$

Example 6.21. Consider the scalar field f defined for all $(x, y, z) \neq (0, 0, 0)$ as

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r}$$

where $r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$ and $\boldsymbol{r} = (x, y, z)$. Show that $\nabla^2 f = 0$.

Solution. The first derivatives are

$$\frac{\partial f}{\partial x} = \frac{3x^2}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \frac{\partial f}{\partial y} = \frac{3y^2}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = \frac{3z^2}{(x^2 + y^2 + z^2)^{3/2}}.$$

Computing the second derivatives, we find that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\ \text{and } \frac{\partial^2 f}{\partial z^2} &= \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned}$$

Thus,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}$$
$$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0.$$

Remark 6.22. We can also take second derivatives of vector fields. Since $\nabla \times \boldsymbol{F}$ is another vector field, we can take the divergence and the curl of $\nabla \times \boldsymbol{F}$. We already know that the divergence of the curl of a vector field is zero:

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0. \tag{6.8}$$

A common identity for the curl of the curl of a vector field is:

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F} \tag{6.9}$$

where $\nabla^2 \boldsymbol{F}$ is the *vector Laplacian* of $\boldsymbol{F}$ defined as

$$\nabla^2 \boldsymbol{F} = \left(\nabla^2 F_1,\, \nabla^2 F_2,\, \nabla^2 F_3\right).$$

6.4.3 Radial vector fields

Recall that a vector field $\boldsymbol{F}$ is said to be *radial* if there is a function f such that

$$\boldsymbol{F}(\boldsymbol{r}) = f(r)\boldsymbol{r}$$

for all $\boldsymbol{r} = (x, y, z)$, where $r = \|\boldsymbol{r}\| = \sqrt{x^2+y^2+z^2}$. Here we will find expressions for the divergence and curl of radial vector fields. First note that we can think of $\boldsymbol{r}$ as a vector field

$$\boldsymbol{r}(x, y, z) = (x, y, z),$$

and thus the divergence of $\boldsymbol{r}$ is

$$\nabla \cdot \boldsymbol{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Furthermore, the curl of $\boldsymbol{r}$ is

$$\nabla \times \boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}.$$

We can also think of r as a scalar field $r(x, y, z) = \sqrt{x^2+y^2+z^2}$, whose gradient is

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial r}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial r}{\partial z}\hat{\boldsymbol{\jmath}} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\imath}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\jmath}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{k}} \\ &= \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z) = \frac{1}{r}\boldsymbol{r}.\end{aligned} \tag{6.10}$$

Now the gradient of the function $f(r) = f(\sqrt{x^2+y^2+z^2})$ can be computed:

$$\begin{aligned}\nabla f(r) &= \frac{\partial}{\partial x} f(r)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y} f(r)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z} f(r)\hat{\boldsymbol{k}} \\ &= f'(r)\frac{\partial r}{\partial x}\hat{\boldsymbol{\imath}} + f'(r)\frac{\partial r}{\partial y}\hat{\boldsymbol{\jmath}} + f'(r)\frac{\partial r}{\partial z}\hat{\boldsymbol{k}} \\ &= f'(r)\left(\frac{\partial r}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial r}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}}\right) \\ &= f'(r)\nabla r \\ &= \frac{f'(r)}{r}\boldsymbol{r},\end{aligned}$$

where we use the fact that $\nabla r = \boldsymbol{r}/r$ from (6.10). Using the product rule in (6.5), the divergence of a radial vector field is

$$\begin{aligned}\nabla \cdot (f(r)\boldsymbol{r}) &= \nabla f(r) \cdot \boldsymbol{r} + f(r)\nabla \cdot \boldsymbol{r} \\ &= \left(\frac{f'(r)}{r}\boldsymbol{r}\right) \cdot \boldsymbol{r} + 3f(r) \\ &= rf'(r) + 3f(r),\end{aligned}$$

where we use the fact that $\boldsymbol{r} \cdot \boldsymbol{r} = \|\boldsymbol{r}\|^2 = r^2$. On the other hand, using the product rule in (6.4), the curl of a radial vector field is

$$\begin{aligned}\nabla \times (f(r)\boldsymbol{r}) &= (\nabla f(r)) \times \boldsymbol{r} + f(r)(\underbrace{\nabla \times r}_{=\mathbf{0}}) \\ &= \frac{f'(r)}{r}\underbrace{\boldsymbol{r} \times \boldsymbol{r}}_{=\mathbf{0}} \\ &= \mathbf{0},\end{aligned}$$

where we use the fact that the cross product of any vector with itself is the zero vector. Hence the divergence and curl of a radial vector field are

$$\nabla \cdot (f(r)\boldsymbol{r}) = rf'(r) + 3f(r) \qquad \text{and} \qquad \nabla \times (f(r)\boldsymbol{r}) = \mathbf{0}. \tag{6.11}$$

Remark 6.23. We can use (6.11) to find conditions for which a radial vector field is conservative. Recall that a vector field $\boldsymbol{F}$ is conservative if $\nabla \cdot \boldsymbol{F} = 0$. Hence, for a radial vector field $\boldsymbol{F}(\boldsymbol{r}) = f(r)\boldsymbol{r}$ to be conservative, it must satisfy

$$rf'(r) + 3f(r) = 0 \qquad \text{or} \qquad \frac{f'(r)}{f(r)} = -\frac{3}{r} \qquad \Longleftrightarrow \qquad \frac{1}{f}\frac{df}{dr} = -\frac{3}{r}$$

We can solve the separable differential equation $f'/f = -3/r$ by integrating both sides

$$\int \frac{1}{f}\frac{df}{dr}\,dr = -3\int \frac{1}{r}\,dr \qquad \Longrightarrow \qquad \ln f(r) = -3\ln r + c_0 \qquad \Longrightarrow \qquad f(r) = cr^{-3}$$

where c_0 is a constant and $c = e^{c_0}$. It follows that a radial vector field $\boldsymbol{F}(\boldsymbol{r}) = f(r)\boldsymbol{r}$ is conservative if and only if f is of the form

$$f(r) = \frac{c}{r^3} \qquad \text{or} \qquad f(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{3/2}}$$

for some constant c.

6.5 The basic laws of Electromagnetism

We have now reached the point where we are ready to put our tools of vector calculus to work and use it to study electromagnetism. The whole point of what we have done so far is to understand the main theorems of vector calculus:

- Fundamental Theorem of Line Integrals: For a C^1-scalar field Ψ and an oriented curve Γ parameterized by $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^3$, it holds that

$$\int_\Gamma \nabla\Psi \cdot d\boldsymbol{r} = \Psi(\boldsymbol{\gamma}(b)) - \Psi(\boldsymbol{\gamma}(a)).$$

- Stokes' Theorem: For a C^1-vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ and a surface $\Sigma \subset \Omega \subseteq \mathbb{R}^3$, it holds that

$$\oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{A}.$$

- Gauss- Ostrogradskii Theorem: For a C^1-vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ on a region $\Omega \subseteq \mathbb{R}^3$, it holds that

$$\oiint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{A} = \iiint_\Omega \nabla \cdot \boldsymbol{F}\, dV.$$

It is not an exaggeration to say that *all* of the basic laws of engineering physics are consequences of these mathematical statements. Here we will primarily be interested in the vector fields $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{J}$ that model the *electric field*, *magnetic field*, and *current density* resulting from electric charges and magnets moving around in space. We also use ρ to denote scalar field of the *charge density* of charges in space. Thus far, we have only concerned ourselves with fields that vary only in space, but in physics we are interested in modeling fields that can also change *time*. Hence we will think of these fields as functions of *four* variables

$$\boldsymbol{E}(x, y, z, t), \qquad \boldsymbol{B}(x, y, z, t), \qquad \boldsymbol{J}(x, y, z, t), \qquad \text{and} \qquad \rho(x, y, z, t)$$

that represent physical quantities in all points in space and at all times. Electric charges have been found experimentally to move in response to forces that are related to these vector fields. The main goal of the study of electromagnetism is to model the movement of charges by finding time-varying fields $\boldsymbol{E}(x, y, z, t)$, $\boldsymbol{B}(x, y, z, t)$, $\boldsymbol{J}(x, y, z, t)$, and $\rho(x, y, z, t)$. As we shall see, not all functions represent physical situations. Only fields that satisfy certain equations can represent physical situations. In fact, these fundamental fields are linked by a set of equations such that their solution gives a complete description of both electrical and magnetic phenomena.

6.5.1 Maxwell's equations in integral form

We begin by stating the *integral form* of Maxwell's equations. This collection of only four equations completely dictates the behaviour of electric and magnetic phenomena.

We must first introduce a few physical constants. In this course, we will primarily be concerned with electromagnetic phenomena in *vacuum*, although in future courses you will work with electromagnetism in different "media." The values of these constants will usually depend on what medium you are dealing with (e.g., air or metal).

- The *permittivity of free space* (i.e., of vacuum) or the *electric constant* is

$$\varepsilon_0 \approx 8.85 \times 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1},$$

  in units of farads per meter. Vaguely speaking, it is the capability of the vacuum to permit electric field lines, or how well the vacuum forms capacitors.

- The *permeability of free space* or the *magnetic constant* is

$$\mu_0 \approx 1.26 \times 10^{-6}\,\mathrm{H}\cdot\mathrm{m}^{-1}$$

in units of henrys per meter (where henry is the SI derived unit of electrical inductance). Vaguely speaking, it is the capability of the vacuum to permit magnetic field lines, or how well the vacuum forms inductors.

We will now present Maxwell's Equations in what is called "integral form." All electric and magnetic fields that arise in physical situations must satisfy this set of equations. These mathematical "laws" were determined experimentally. We will discuss them in due course.

Maxwell's equations in vacuum (in integral form):

- **Gauss' Law**: The total electric flux through a closed surface is proportional to the total amount of charge enclosed inside that surface.

$$\boxed{\oiint_{\partial\Omega} \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{Q_{\text{encl}}}{\varepsilon_0}} \tag{6.12}$$

- **Faraday's Law**: The circulation of the electric field around a closed loop is proportional to the rate of change of the magnetic flux through the loop.

$$\boxed{\oint_{\partial\Sigma} \boldsymbol{E}\cdot d\boldsymbol{r} = -\frac{\partial}{\partial t}\iint_{\Sigma} \boldsymbol{B}\cdot d\boldsymbol{A}} \tag{6.13}$$

- **No magnetic monopoles** (Gauss's Law for magnetism): The total magnetic flux through any closed surface is zero.

$$\boxed{\oiint_{\partial\Omega} \boldsymbol{B}\cdot d\boldsymbol{A} = 0} \tag{6.14}$$

- **Ampère's Law**: The circulation of the magnetic field around a closed loop is proportional to the rate of change of the electric flux plus the current through the loop.

$$\boxed{\oint_{\partial\Sigma} \boldsymbol{B}\cdot d\boldsymbol{r} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_{\Sigma} \boldsymbol{E}\cdot d\boldsymbol{A} + \mu_0\iint_{\Sigma} \boldsymbol{J}\cdot d\boldsymbol{A}} \tag{6.15}$$

6.5.2 Gauss' Law

If a large point charge Q is placed at the origin and another smaller point charge q is placed at some point $\boldsymbol{r} \in \mathbb{R}^3$ in space, it has been determined experimentally that the *force* felt by the small charge is precisely equal to

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{kQq}{r^3}\boldsymbol{r}$$

where $k = \frac{1}{4\pi\varepsilon_0}$ is the *electric constant.* Mathematically, it is easier to introduce the concept of an "electric field" rather than dealing directly with the force, such that the force felt by the charge q is given by

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{kQq}{r^3}\boldsymbol{r} = q\boldsymbol{E}(\boldsymbol{r}) \qquad \text{where} \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{kQ}{r^3}\boldsymbol{r}.$$

We can now verify Gauss' Law for point-particles. Consider the field

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{kQ}{r^3}\boldsymbol{r}$$

emanating from a point particle with charge Q placed at the origin. For a fixed value $R > 0$, we can consider the surface Σ_R of the sphere of radius R centered at the origin with outward pointing normal $\hat{\boldsymbol{n}} = \boldsymbol{r}/\|\boldsymbol{r}\|$ at each point on the surface. The dot product of the field with the normal at each point on the surface is

$$\boldsymbol{E}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = \frac{kQ}{r^3}\boldsymbol{r} \cdot \left(\frac{\boldsymbol{r}}{r}\right) = \frac{kQ}{R^4}(\boldsymbol{r} \cdot \boldsymbol{r}) = \frac{kQ}{R^2}$$

since $\boldsymbol{r} \cdot \boldsymbol{r} = \|\boldsymbol{r}\|^2 = R^2$ at each point on Σ_R. Computing the flux of the electric field through this surface is therefore

$$\oiint_{\Sigma_R} \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA = \oiint_{\Sigma_R} \frac{kQ}{R^2}\, dA = \frac{kQ}{R^2}\operatorname{area}(\Sigma_R) = 4\pi kQ = \frac{Q}{\varepsilon_0},$$

where we recall that $k = 1/(4\pi\varepsilon_0)$, and we find that

$$\frac{1}{\varepsilon_0}Q = \oiint_{\Sigma_R} \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA.$$

Practically speaking, it is often convenient to model charges as infinitessimally small points. When modelling electric phenomena, however, it is better to model charge *distributions*. Suppose $\rho : \mathbb{R}^3 \to \mathbb{R}$ is the scalar field indicating the *charge density* at points in space. Then the total amount of charge contained inside a region $\Omega \subset \mathbb{R}^3$ is equal to

$$\text{total charge contained in } \Omega = \iiint_\Omega \rho\, dV.$$

Gauss' Law (i.e., Maxwell's first equation) can now be rewritten as follows:

$$\boxed{\oiint_{\partial\Omega} \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{1}{\varepsilon_0}\iiint_\Omega \rho\, dV} \tag{6.16}$$

However, we can use the Divergence Theorem of Gauss-Ostogradskii to rewrite the flux integral on the left-hand side as a volume integral

$$\oiint_{\partial\Omega} \boldsymbol{E} \cdot d\boldsymbol{A} = \iiint_\Omega \nabla \cdot \boldsymbol{E}\, dV.$$

Substituting this into (6.16) and rearranging yields

$$\iiint_\Omega \nabla \cdot \boldsymbol{E}\, dV = \frac{1}{\varepsilon_0}\iiint_\Omega \rho\, dV \qquad \text{or} \qquad \iiint_\Omega \left(\nabla \cdot \boldsymbol{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0.$$

We now need a useful mathematical fact.

Theorem 6.24 (du Bois-Reymond Lemma, Part I)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^0-scalar field. If it holds that*

$$\iiint_\Omega f\, dV = 0$$

for all possible choices of region $\Omega \subset \mathbb{R}^3$, then $f = 0$.

The *du Bois-Reymond Lemma* effectively says that, if integrating a continuous scalar field over any choice of solid region always is zero, then the scalar field itself must have been zero. This makes intuitive sense, so we will not delve into this any deeper. However, we can now use this to see the following fact:

$$\iiint_\Omega \left(\nabla \cdot \boldsymbol{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0 \text{ for all possible regions } \Omega \qquad \Longrightarrow \qquad \nabla \cdot \boldsymbol{E} - \frac{\rho}{\varepsilon_0} = 0,$$

or equivalently

$$\boxed{\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}} \tag{6.17}$$

The equation in (6.17) is Gauss' Law in "differential form." An analogous argument shows that

$$\boxed{\nabla \cdot \boldsymbol{B} = 0.} \tag{6.18}$$

That is, "magnetic fields are always solenoidal (divergence-free)."

6.5.3 Faraday's Law and Ampère's Law

A similar fact can be used to derive "differential forms" of Faraday's Law and Ampère's Law. This is another version of the du Bois-Reymond Lemma, but for surface integrals.

Theorem 6.25 (du Bois-Reymond Lemma, Part II)**.** *Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^0-vector field. If it holds that*

$$\iint_\Sigma \boldsymbol{F} \cdot d\boldsymbol{A} = 0$$

for all possible choices of surface $\Sigma \subset \mathbb{R}^3$, then $\boldsymbol{F} = \boldsymbol{0}$.

This version of the du Bois-Reymond Lemma says that, if integrating a continuous vector field over any choice of surface in space is always is zero, then the vector field itself must have been zero. We can now use this to derive differential versions of the rest of Maxwell's equations.

Note that we can use Stokes' Theorem to convert the circulation integral in Faraday's Law to a surface integral:

$$\oint_{\partial\Sigma} \boldsymbol{E} \cdot d\boldsymbol{r} = \iint_\Sigma (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A}.$$

Inserting this into the integral version of Faraday's Law and rearranging yields:

$$\iint_\Sigma (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A} = -\frac{\partial}{\partial t} \iint_\Sigma \boldsymbol{B} \cdot d\boldsymbol{A} \qquad \Longrightarrow \qquad \iint_\Sigma \left((\nabla \times \boldsymbol{E}) - \frac{\partial}{\partial t}\boldsymbol{B}\right) \cdot d\boldsymbol{A} = 0.$$

The second version of the du Bois-Reymond Lemma can now be used to deduce the following fact:

$$\iint_\Sigma \left(\nabla \times \boldsymbol{E} - \frac{\partial \boldsymbol{B}}{\partial t}\right) \cdot d\boldsymbol{A} = 0 \text{ for all possible surfaces } \Sigma \qquad \Longrightarrow \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}.$$

This gives us the "differrential version" of Faraday's Law.

$$\boxed{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \tag{6.19}$$

An analogous argument can be used to find the differential version of Ampère's Law:

$$\boxed{\nabla \times \boldsymbol{B} = \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J}}. \tag{6.20}$$

6.5.4 Maxwell's Equations (differential version)

We can now collect all of the differential versions and present them in one place:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{6.21}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{6.22}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.23}$$

$$\nabla \times \boldsymbol{B} = \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} \tag{6.24}$$

After the midterm we will get some practice working with these equations and understanding them.