ECE 206 – University of Waterloo **Fall 2019**

Lecture notes for Week 10

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10.1 Cauchy-Riemann Equations in polar form

Recall that a mapping of the form $f(x+jy) = u(x,y) + jv(x,y)$ is differentiable at a point z_0 if and only if the functions u and v are differentiable as functions of real numbers and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{10.1}$$

at $z_0 = x_0 + jy + 0$. Moreover, if u and v satisfy (10.1) at $z_0 = x_0 + jy + 0$ then the derivative of f as a mapping of complex numbers is given by

$$f'(z_0) = u_x(x_0, y_0) + jv_x(x_0, y_0) \qquad \text{or} \qquad f'(z_0) = v_y(x_0, y_0) - ju_y(x_0, y_0),$$

where u_x, v_x, u_y, and v_y are the partial derivatives of u and v.

Not all complex mappigns can conveniently be expressed in terms of the Cartesian coordinates for the input. Since every complex number can be written as $z = re^{j\theta}$ it is also common to express a function in *polar form* as

$$f(re^{j\theta}) = u(r,\theta) + jv(r,\theta)$$

where u and v still represent the real and imaginary components of $f(z)$, but are now functions of the modulus and the argument of z rather than the real and imaginary components. For example, the logarithm of a complex number can be expressed as

$$\operatorname{Log}(re^{j\theta}) = \ln r + j\theta$$

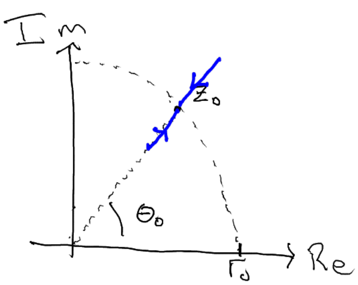
assuming $\theta \in (-\pi, \pi]$ is the principal argument, which has real and imaginary components $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. For functions like this, there is a different form of the Cauchy-Riemann Equations that is more useful, which we will derive here.

Suppose that $f : D \to \mathbb{C}$ is a mapping that is differentiable at a point $z_0 \in D$. This imples that the limit

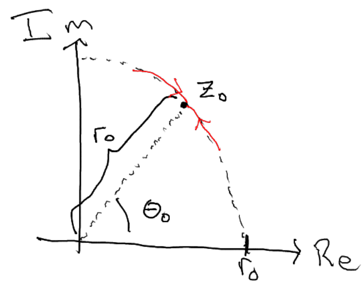
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{10.2}$$

exists, since we have assumed that f is differentiable at z_0, and we must obtain the same value of the limit no matter what path we use to apprach z_0. As we did previously for finding the Cauchy-Riemann Equations in Cartesian form, we will consider now two of the possible ways to approach z_0: first by varying r and holding $\theta = \theta_0$ constant, then by varying θ and holding $r = r_0$ constant.

(i) First consider the approach $z \to z_0$ where we hold $\theta = \theta_0$ constant and vary r such that $z = re^{j\theta_0}$ (see Figure 10.1a). Expanding out f in terms of its real and imaginary components as functions of r and



(a) Approaching z_0 along the path with $\theta = \theta_0$ held constant and varying r.



(b) Approaching z_0 along the path with $r = r_0$ held constant and varying θ.

Figure 10.1: To derive the Cauchy-Riemann equations in polar form, we consider two ways of approaching a fixed point z_0.

θ, the limit in (10.2) becomes

$$\begin{aligned}
f'(z_0) = \lim_{r\to r_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{r\to r_0} \frac{f(re^{j\theta_0}) - f(r_0 e^{j\theta_0})}{re^{j\theta_0} - r_0 e^{j\theta_0}} \\
&= \frac{1}{e^{j\theta_0}} \lim_{r\to r_0} \frac{u(r,\theta_0) + jv(r,\theta_0) - (u(r_0,\theta_0) + jv(r_0,\theta_0))}{r - r_0} \\
&= e^{-j\theta_0} \left(\lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r - r_0} + j \lim_{r\to r_0} \frac{v(r,\theta_0) - v(r_0,\theta_0)}{r - r_0} \right) \\
&= e^{-j\theta_0} \left(\left.\frac{\partial u}{\partial r}\right|_{(r_0,\theta_0)} + j \left.\frac{\partial v}{\partial r}\right|_{(r_0,\theta_0)} \right) \\
&= e^{-j\theta_0} \big(u_r(r_0,\theta_0) + jv_r(r_0,\theta_0) \big) \qquad (10.3)
\end{aligned}$$

(ii) On the other hand, consider the approach $z \to z_0$ where we hold $r = r_0$ constant and vary θ such that $z = r_0 e^{j\theta}$ (see Figure 10.1b). Expanding out f in terms of its real and imaginary components as functions of r and θ, the limit in (10.2) becomes

$$\begin{aligned}
f'(z_0) &= \lim_{\theta\to\theta_0} \frac{f(r_0 e^{j\theta}) - f(r_0 e^{j\theta_0})}{r_0 e^{j\theta} - r_0 e^{j\theta_0}} \\
&= \frac{1}{r_0} \lim_{\theta\to\theta_0} \frac{u(r_0,\theta) + jv(r_0,\theta_0) - (u(r_0,\theta_0) + jv(r_0,\theta_0))}{e^{j\theta} - e^{j\theta_0}} \\
&= \frac{1}{r_0} \lim_{\theta\to\theta_0} \left(\frac{u(r_0,\theta) - u(r_0,\theta_0) + jv(r_0,\theta_0) - jv(r_0,\theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{j\theta} - e^{j\theta_0}} \right) \\
&= \frac{1}{r_0} \left(\lim_{\theta\to\theta_0} \frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + j \lim_{r\to r_0} \frac{v(r,\theta_0) - v(r_0,\theta_0)}{r - r_0} \right) \Bigg(\underbrace{\lim_{\theta\to\theta_0} \frac{\theta - \theta_0}{e^{j\theta} - e^{j\theta_0}}}_{= -je^{-j\theta_0}} \Bigg) \qquad (10.4)
\end{aligned}$$

where the value of the rightmost limit in (10.4) can be found by

$$\begin{aligned}
\lim_{\theta\to\theta_0} \frac{\theta - \theta_0}{e^{j\theta} - e^{j\theta_0}} &= \left(\lim_{\theta\to\theta_0} \frac{e^{j\theta} - e^{j\theta_0}}{\theta - \theta_0} \right)^{-1} \\
&= \left(\lim_{\theta\to\theta_0} \frac{f(\theta) - f(\theta_0)}{\theta - \theta_0} \right)^{-1} = \big(f'(\theta_0)\big)^{-1} = \big(je^{j\theta_0}\big)^{-1} = -je^{-j\theta_0}
\end{aligned}$$

and we make use of the function $f(\theta) = e^{j\theta}$ which has derivative $f'(\theta) = je^{j\theta}$. Continuing from (10.4), we can express $f'(z_0)$ as

$$\begin{aligned} f'(z_0) &= \frac{1}{r_0}\left(\left.\frac{\partial u}{\partial \theta}\right|_{(r_0,\theta_0)} + j\left.\frac{\partial v}{\partial \theta}\right|_{(r_0,\theta_0)}\right)\left(-je^{-j\theta_0}\right) \\ &= \frac{e^{-j\theta_0}}{r_0}\left(\left.\frac{\partial v}{\partial \theta}\right|_{(r_0,\theta_0)} - j\left.\frac{\partial u}{\partial \theta}\right|_{(r_0,\theta_0)}\right) \\ &= \frac{e^{-j\theta_0}}{r_0}\big(v_\theta(r_0,\theta_0) - ju_\theta(r_0,\theta_0)\big). \end{aligned} \tag{10.5}$$

Considering that it should not matter which method we use to approach $z_0 = e_0 e^{j\theta_0}$ for the limit in (10.2) to exist, the values of these two limits must be equal. Equating lines (10.3) and (10.5), we see that if f is differentiable then it must be the case that

$$f'(r_0 e^{j\theta_0}) = e^{-j\theta_0}\big(u_r(r_0,\theta_0) + jv_r(r_0,\theta_0)\big) = \frac{e^{-j\theta_0}}{r_0}\big(v_\theta(r_0,\theta_0) - ju_\theta(r_0,\theta_0)\big). \tag{10.6}$$

Removing the the common factor of $e^{-j\theta}$ and equating the real and imaginary parts of both sides, we find that

$$u_r(r_0,\theta_0) = \frac{v_\theta(r_0,\theta_0)}{r_0} \qquad \text{and} \qquad v_r(r_0,\theta_0) = -\frac{u_\theta(r_0,\theta_0)}{r_0}.$$

This gives us the *Cauchy-Riemann equations in polar form*:

$$\boxed{u_r - \frac{1}{r}v_\theta = 0 \qquad \text{and} \qquad v_r + \frac{1}{r}u_\theta = 0.} \tag{10.7}$$

We have shown that, if $f(re^{j\theta}) = u(r,\theta) + jv(r,\theta)$ is complex-differentiable at a point $z_0 = r_0 e^{j\theta_0}$, then the functions u and v must satisfy (10.7). It turns out that the reverse implication is true as well. This fact is useful for showing that certain functions are differentiable, but understanding its proof is beyond the scope of this course.

Theorem 10.1. *Suppose $f : D \to \mathbb{C}$ is a mapping on some region $D \subseteq \mathbb{C}$ with real and imaginary components given by differentiable functions u and v such that*

$$f(re^{j\theta}) = u(r,\theta) + jv(r,\theta).$$

It holds that f is differentiable at a poinnt $z_0 = r_0 e^{j\theta_0}$ if and only if the functions u and v satisfy the Cauchy-Riemann equations in polar form (10.7) *at* (r_0,θ_0).

Moreover, if a mapping $f(re^{j\theta}) = u(r,\theta) + jv(r,\theta)$ is differentiable at a point $z_0 = r_0 e^{j\theta_0}$, its derivative at that point is given by either expression in (10.6).

Example 10.2. The principal logarithm is a good example of a function that is best expressed in polar form. For a complex number $z = re^{j\theta}$ with $r > 0$ and $-\pi < \theta < \pi$, we have

$$\mathrm{Log}(z) = \mathrm{Log}(re^{j\theta}) = \ln r + j\theta = u(r,\theta) + jv(r,\theta)$$

where the real and imaginary parts are $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. These functions satisfy the Cauchy-Riemann equations in polar form, since

$$u_r - \frac{1}{r}v_\theta = \frac{\partial}{\partial r}\ln r - \frac{1}{r}\frac{\partial}{\partial \theta}\theta = \frac{1}{r} - \frac{1}{r} = 0.$$

Moreover, the derivative of $f(z) = \text{Log}(z)$ is given by

$$f'(z_0) = e^{-j\theta_0}\big(u_r(r_0, \theta_0) + jv_r(r_0, \theta_0)\big) = \frac{1}{e^{j\theta_0}}\left(\frac{1}{r_0} - 0\right) = \frac{1}{r_0 e^{j\theta_0}} = \frac{1}{z_0}.$$

This confirms what was stated last week, which is that the derivative of $\text{Log}\, z$ is $1/z$, as we should expect from calculus.

10.2 Harmonic functions

The study of *harmonic functions* is very important in developing solutions to boundary-value problems of differential equations in many areas of physics and engineering, including electromagnetism. The name "harmonic" originates from the harmonic motion that a point on a vibrating string undergoes. Solutions to differential equations for this type of motion can be written in terms of sines and cosines. Fourier analysis involves expanding periodic functions in terms of a series of these harmonics. While the applications of harmonic functions will not be the focus of this course, we will investigate how the idea of harmonic functions relates to complex analysis.

Definition 10.3. A C^2 function $f : D \to \mathbb{R}$ on a region $D \subseteq \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{10.8}$$

everywhere in D.

The differential equation (10.8) is often called *Laplace's equation*, and can be rewritten as $f_{xx} + f_{yy} = 0$ or $\nabla^2 f = 0$, where ∇^2 is the Laplace operator defined previously in this course

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

(Here the Laplace operator is defined in two dimensions, although it can be defined for functions of any number of dimensions.) Harmonic functions are solutions to Laplace's equation.

We'll now explore how the study of harmonic functions relates to complex analysis. Let $f : D \to \mathbb{C}$ be a mapping that is differentiable everywhere in $D \subseteq \mathbb{C}$. We can consider D as a subset of $\mathbb{R}^2$ as well. If f has real and imaginary components given by C^2 functions u and v,

$$f(x + jy) = u(x, y) + jv(x, y),$$

then u and v must be harmonic! Indeed, note that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = v_{yx} - v_{xy} = 0,\end{aligned}$$

where $u_x = v_y$ and $u_y = -v_x$ follow from the fact that u and v must satisfy the Cauchy-Riemann equations (10.1), and $v_{yx} = v_{xy}$ since we have assumed v to be C^2. It can be shown that v is harmonic as well.

Definition 10.4. Two harmonic C^2 functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ are said to be *harmonic conjugates* if together they satisfy the Cauchy-Riemann equations (10.1).

In particular we see that any pair of harmonic functions are harmonic conjugates if and only if they can be viewed as the real and imaginary parts of some complex-differentiable mapping $f(x+jy) = u(x, y)+jv(x, y)$. Given one harmonic function $u : \mathbb{R}^2 \to \mathbb{R}$, a harmonic conjugate to u can always be found, as the following example shows.

Example 10.5. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $u(x, y) = 2x(1 - y)$ for all $x, y \in \mathbb{R}$. Show that u is harmonic and find a harmonic conjugate v for u. Find a mapping $f : \mathbb{C} \to \mathbb{C}$ whose real and imaginary components are given by u and v.

Solution. The derivatives of u with respect to x and y are $u_x = 2(1 - y)$ and $u_y = -2x$, and we see that $u_{xx} = u_{yy} = 0$, so u is indeed harmonic. To find a harmonic conjugate for u, we must find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that u and v jointly satisfy the Cauchy-Riemann equations

$$v_y = u_x \qquad \text{and} \qquad v_x = -u_y.$$

Integrating both sides of the equation $v_y = u_x$ with respect to y gives us a form for v, and thus v must have the form

$$\begin{aligned} v(x, y) = \int v_y \, dy &= \int u_x \, dy \\ &= \int 2(1 - y) \, dy = 2y - y^2 + g(x) \end{aligned}$$

for some function g whose value is dependent only on x (i.e., constant with respect to y). On the other hand, integrating both sides of the equation $v_x = -u_y$ with respect to x gives us another form for v,

$$\begin{aligned} v(x, y) = \int v_x \, dx &= -\int u_y \, dx \\ &= -\int -2x \, dx = x^2 + h(y) \end{aligned}$$

for some function h whose value is dependent only on y. Comparing these two forms for v, we see that any harmonic conjugate of u must be

$$v(x, y) = x^2 + 2y - y^2 + c$$

for some constant $c \in \mathbb{R}$. Let us now find the form of a mapping $f : \mathbb{C} \to \mathbb{C}$ such that $f(x + jy) = u(x, y) + jv(x, y)$. That is,

$$\begin{aligned} f(x + jy) &= 2x(1 - y) + j(x^2 + 2y - y^2) \\ &= 2x + j2y + jx^2 - 2xy - jy^2 = 2(x + jy) + j(\underbrace{x^2 + j2xy - y^2}_{(x+jy)^2}) = 2(x + jy) + j(x + jy)^2, \end{aligned}$$

from which we see that f is the mapping defined by $f(z) = z + jx^2 = z(1 + jz)$ for all $z \in \mathbb{C}$.

Functions u and v that are harmonic conjugates also have a nice geometric interpretation. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ are harmonic conjugates and consider the gradients ∇u and ∇v of each,

$$\nabla u = (u_x, u_y) \qquad \text{and} \qquad \nabla v = (v_x, v_y).$$

The Cauchy-Riemann equations imply that ∇u and ∇v must *always be orthogonal to each other*

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = v_y v_x - (v_x) v_y = 0,$$

since the Cauchy-Riemann equations imply that $u_x = v_y$ and $u_y = -v_x$. What does this mean exactly? Consider now the *level curves* of u and v, where a level curve of u is the set of points (x, y) satisfying the equation $u(x, y) = c$ for some constant $c \in \mathbb{R}$. How are the level curves of u related to the gradient of u? Well, the level curves are always orthogonal to the gradient of u. Indeed, suppose some level curve

$$\Gamma_c = \{(x, y) \,|\, u(x, y) = c\}$$

is parameterized by some path $\boldsymbol{\gamma}(t) = (x(t), y(t))$. Differentiating both sides of the equation $c = u(x(t), y(t))$ will yield zero (since differentiating a constant is always zero) and thus

$$0 = \frac{dc}{dt} = \frac{d}{dt} u(x(t), y(t)) = u_x\big(x(t), y(t)\big) x'(t) + u_y\big(x(t), y(t)\big) y'(t) = \nabla u\big(x(t), y(t)\big) \cdot \boldsymbol{\gamma}'(t).$$

Hence the level curve of u is everywhere orthogonal to the gradient of u, since $\nabla u \cdot \boldsymbol{\gamma}' = 0$.

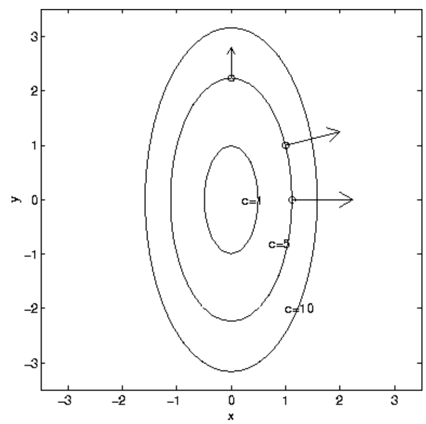


Figure 10.2: The level curves of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are orthogonal to the gradient of u at each point. This figure shows some level curves of the function $f(x, y) = 4x^2 + y^2$ and the gradient at a few points.

Since the gradients ∇u and ∇v are orthogonal to the level curves of u and v respectively, it follows that the level curves of u are orthogonal to the level curves of v! That is, if u and v are harmonic conjugates, the level curves of u must always intersect perpendicularly with the level curves of v.

Example 10.6. Consider the functions $u(x, y) = 2x(1 - y)$ and $v(x, y) = x^2 + 2y - y^2$ from Example 10.5, which we have shown to be harmonic conjugates. The surfaces of the graphs of u and v are shown in Figures 10.3 and 10.4 (where the surfaces in Figure 10.4 show the level curves of u and v drawn onto the graphs). Superimposing the sketches of the level curves of u and v together on the plane yields the graphic in Figure 10.5, which shows that the level curves of u and v always intersect perpendicularly.

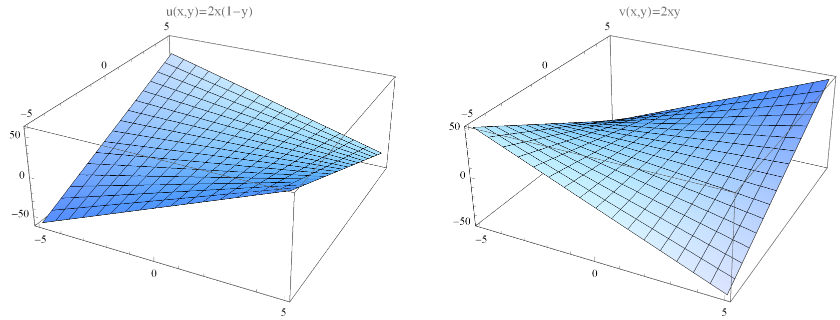


Figure 10.3: Plots of the graphs of $u(x, y) = 2x(1 - y)$ and $v(x, y) = x^2 + 2y - y^2$.

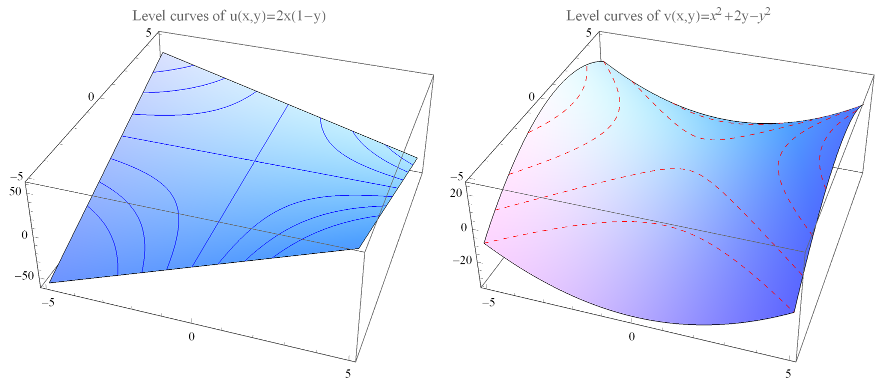


Figure 10.4: Plots of the graphs of $u(x, y) = 2x(1 - y)$ and $v(x, y) = x^2 + 2y - y^2$ with some of the level curves shown on the surfaces of the graphs.

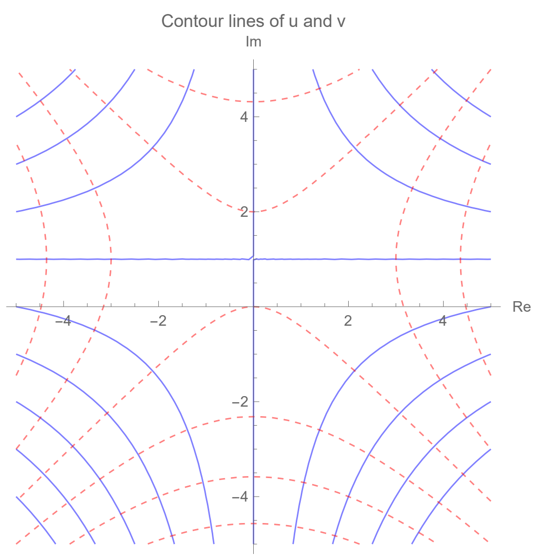


Figure 10.5: Some of the level curves of u (blue) and v (dashed red). Note that u and v are harmonic conjugates, so their level curves always intersect perpendicularly.

10.3 Integration of complex functions

10.3.1 Complex-valued functions with real inputs

Consider a function $f : [a, b] \to \mathbb{C}$ that takes real numbers in the interval $[a, b] \subset \mathbb{R}$ as inputs and outputs complex numbers. Any such function can be separated out into its real and complex components

$$f(t) = u(t) + jv(t)$$

for some functions $u, v : [a, b] \to \mathbb{R}$. Such a function f is said to be differentiable if each of its components u and v are differentiable, and the derivative of f is defined as

$$f'(t) = u'(t) + jv'(t),$$

and the function f is said to be C^1 if each of its components u and v is a C^1 function of real variables. We can also integrate these kinds of complex-valued functions by integrating the real and imaginary parts separately,

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + j\int_a^b v(t)\,dt.$$

Example 10.7. Consider the function $f(t) = e^{jt}$, which can be written as $f(t) = \cos t + j \sin t$. Integrating this over some interval $[a, b]$, we have

$$\begin{aligned}\int_a^b e^{jt}\,dt &= \int_a^b \cos t\,dt + j\int_a^b \sin t\,dt \\ &= \sin t\,\Big|_a^b - j\cos t\,\Big|_a^b \\ &= -j\big(\cos t + j\sin t\big)\Big|_a^b \\ &= \frac{1}{j}e^{jt}\Big|_a^b.\end{aligned}$$

Moreover, if $f : [a, b] \to \mathbb{C}$ is any differentiable function, it holds that

$$\int_a^b f'(t)\,dt = f(b) - f(a),$$

which can be seen by integrating the real parts u' and v' separately.

10.3.2 Contour integration

A *path* in the complex plane is a continuous function $\gamma : [a, b] \to \mathbb{C}$. We are generally only concerned with *differentiable* paths. A path $\gamma : [a, b] \to \mathbb{C}$ in the complex plane with component functions $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$ such that

$$\gamma(t) = \gamma_1(t) + j\gamma_2(t)$$

for all $t \in [a, b]$ is differentiable if γ_1 and γ_2 are differentiable as functions of a real variable, and its derivative is given by $\gamma'(t) = \gamma_1'(t) + j\gamma_2'(t)$. We can integrate along differentiable paths.

Suppose $f : D \to \mathbb{C}$ is a complex mapping on a region $D \subset \mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a differentiable path such that $\gamma(t) \in D$ for all $t \in [a, b]$. The integral of f along the path γ is defined as

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

A path $\gamma : [a, b] \to \mathbb{C}$ is said to be *piecewise differentiable* if we can partition the interval $[a, b]$ in to finitely many subintervals

$$[a_0, a_1],\ [a_1, a_2],\ \cdots,\ [a_{n-1}, a_n] \subset [a, b]$$

for numbers $a_0 < a_1 < a_2 < \cdots < a_n$ with $a = a_0$ and $b = b_0$, such that γ is differentiable on the interior of each subinterval $[a_i, a_{i+1}]$. Likewise, the integral of a complex mapping $f : D \to \mathbb{C}$ along a piecewise differentiable path is defined by integrating along each differential part of the partition

$$\int_{a_0}^{a_1} f(\gamma(t))\gamma'(t)\,dt + \int_{a_1}^{a_2} f(\gamma(t))\gamma'(t)\,dt + \cdots + \int_{a_{n-1}}^{a_n} f(\gamma(t))\gamma'(t)\,dt.$$

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise differentiable path. The *contour* Γ traced out by γ is the oriented curve traced out by γ. The integral of f along the oriented contour is

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Similar to path integrals of vector fields, contour integrals are independent of parameterization (as long as the parameterization is "nice" and doesn't double-back on itself anywhere).

Example 10.8. Consider the mapping $f(z) = z^2$. Integrate f along the contour following contours (see Figure 10.6)

(i) Γ_1 is the line segment from 0 to $1 + j$.

(ii) Γ_2 is the line segment from 0 to 1, followed by the line segment from 1 to $1 + j$.

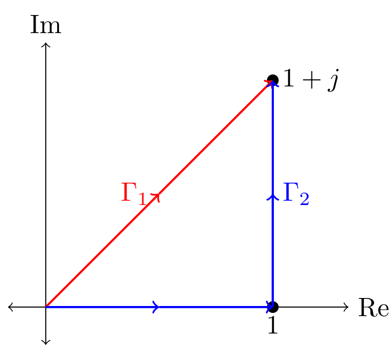


Figure 10.6: Depiction of the oriented contours Γ_1 and Γ_2 example 10.8

Solution. Both contours start at 0 and end at $1 + j$, but traverse along different paths.

(i) This first contour is parameterized by the path $\gamma(t) = t + jt$ for $t \in [0, 1]$, which has derivative $\gamma'(t) = 1 + j$. The desired integral is therefore

$$\begin{aligned}\int_{\Gamma_1} f(z)\,dz &= \int_0^1 f(\gamma(t))\,\gamma'(t)\\ &= \int_0^1 (t + jt)^2(1 + j)\,dt = (1 + j)^3 \int_0^1 t^2\,dt = \frac{(1 + j)^3}{3} t^3\Big|_0^1 = \frac{2(j - 1)}{3}.\end{aligned}$$

(ii) For this contour, we must paramaterize and integrate over the the two segments separately before adding the results together. Along the segment from 0 to 1, the integral is

$$\int_0^1 t^2\,dt = \frac{t}{3}\Big|_0^1 = \frac{1}{3}, \tag{10.9}$$

while the segment from 1 to $1+j$ can be parameterized by $\gamma(t) = 1 + jt$ for $t \in [0,1]$ with derivative $\gamma'(t) = j$. The integral along this segment is

$$\int_0^1 f(\gamma(t))\,\gamma'(t) = \int_0^1 (1+jt)^2 j\,dt = j\int_0^1 (1-t^2+2jt)\,dt = j\left(t - \frac{t^3}{3} + jt^2\right)\Bigg|_0^1 = j\frac{2}{3} - 1. \quad (10.10)$$

Adding together the integrals from (10.9) and (10.10) yields the integral along all of Γ_2, which is

$$\int_{\Gamma_2} z^2\,dz = \frac{1}{3} + j\frac{2}{3} - 1 = \frac{2(j-1)}{3}.$$

Interestingly, the values of the integral of $f(z) = z^2$ along the two different contours Γ_1 and Γ_2 are the same! Is this a coincidence? As we will soon see, this is due to a deeper fact about integrals of complex-differentiable functions.

A contour Γ is said to be *closed* if it is parameterized by a path $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$ (that is, if it starts and ends at the same point). As with path integrals along closed paths in vector calculus, we use a special notation for integrals along closed contours,

$$\oint_\Gamma f(z)\,dz,$$

where the circle on the integral indicates that the contour Γ is closed.

As any contour Γ has an orientation, we may consider what happens if we integrate along the same contour with the *reverse orientation*, indicated by $-\Gamma$, which we expect to be

$$\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz.$$

Indeed, if $\gamma : [a,b] \to \mathbb{C}$ parameterizes Γ, then $-\Gamma$ is parameterized by the path $\beta : [-b,-a] \to \mathbb{C}$ defined as $\beta(s) = \gamma(-s)$ for all $t \in [-b,-a]$ which has derivative given by $\beta'(t) = -\gamma'(-t)$. The integral along the reversed contour is therefore

$$\int_{-\Gamma} f(z)\,dz = \int_{-b}^{-a} f(\beta(t))\beta'(t)\,dt = -\int_{-b}^{-a} f(\gamma(-t))\gamma'(-t)\,dt = -\int_a^b f(\gamma(t))\gamma'(t)\,dt = -\int_\Gamma f(z)\,dz.$$

Any two contours Γ_1 and Γ_2 with the same start and endpoints can therefore be combined into a closed contour $\Gamma = \Gamma_1 \cup (-\Gamma_2)$ that traverses first along Γ_1 then backwards along Γ_2. For example, in Example 10.8, integrating along the closed loop that starts at 0, traverses along the straight line segment Γ_1 to $1+j$, then backwards along the contour Γ_2 to 1 and back to 0, gives us

$$\begin{aligned}\oint_\Gamma f(z)\,dz &= \int_{\Gamma_1} f(z)\,dz + \int_{-\Gamma_2} f(z)\,dz \\ &= \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz = \frac{2(j-1)}{3} - \frac{2(j-1)}{3} = 0,\end{aligned}$$

and thus integrating $f(z) = z^2$ along this particular closed contour yields zero. However, it is not necessarily the case that integrating any function along any closed contour will always be zero, as the following example shows.

Example 10.9. Consider the mapping $f(z) = z^{-1}$ which is defined for all $z \neq 0$ and differentiable everywhere it is defined. Integrate f along a circle of radius $a > 0$ centered at the origin and oriented counter-clockwise.

Solution. A circle centered at the origin is paramtereized by the path $\gamma(t) = ae^{jt}$ for $t \in [0, 2\pi]$, which has given by

$$\gamma'(t) = aje^{jt}.$$

The integral of f along this path is

$$\begin{aligned}\oint_\Gamma \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}\,dt \\ &= \int_0^{2\pi} \frac{aje^{jt}}{ae^{jt}}\,dt = j\int_0^{2\pi}\,dt = 2\pi j.\end{aligned}$$

Note that this is independent of the value of the radius a. In fact, the integral of z^{-1} along *any* closed contour that goes counter-clockwise around the origin will result in the same value.