ECE 206 – University of Waterloo	**Fall 2019**

Lecture notes for Week 11

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11.1 Complex integration of differentiable mappings along closed contours

Here we will discuss methods of integrating differentiable mappings $f : D \to \mathbb{C}$ along closed contours. It will first be helpful to review a few definitions. Analogous to regions in $\mathbb{R}^2$, we say a region $D \subseteq \mathbb{C}$ is *connected* if any two points in D can be connected by a contour in D, and is said to be *simply connected* if it does not have any 'holes' (see Figure 11.1).

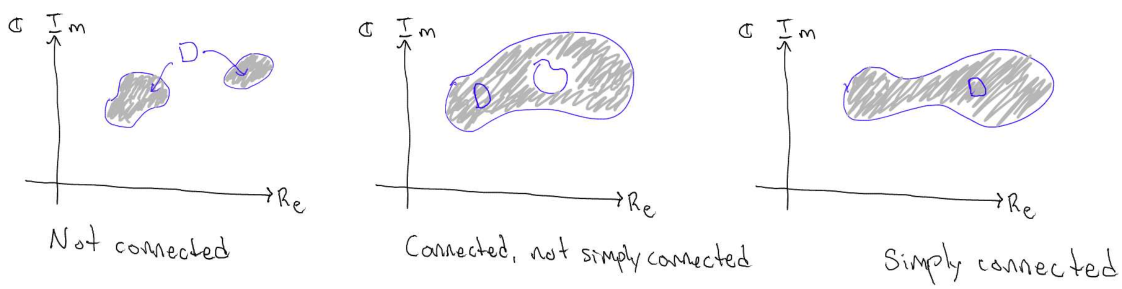

Figure 11.1: Connected and simply connected regions in $\mathbb{C}$.

Similarly, analogous to curves in $\mathbb{R}^2$, we say a contour Γ in $\mathbb{C}$ is *closed* if every parameterization $\gamma : [a, b] \to \mathbb{C}$ of Γ necessarily has $\gamma(a) = \gamma(b)$. Also, a contour is *simple* if it doesn't have any self-intersections. (See Figure 11.2.)

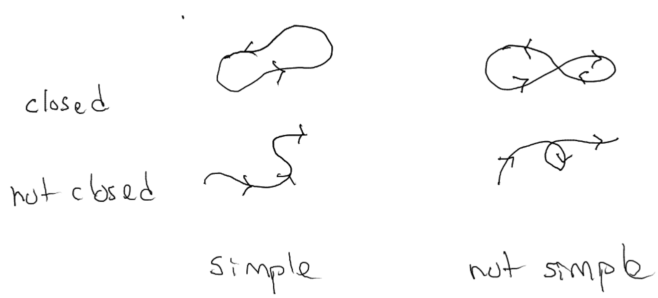

Figure 11.2: Simple and closed contours in $\mathbb{C}$.

Note that any simple closed contour Γ necessarily encloses a simply connected region R in the complex plane. A simple closed contour is said to be *positively oriented* if it goes counter clockwise around the region it encloses. Any such curve can be thought of as the boundary of the region it encloses. (See Figure 11.3.)

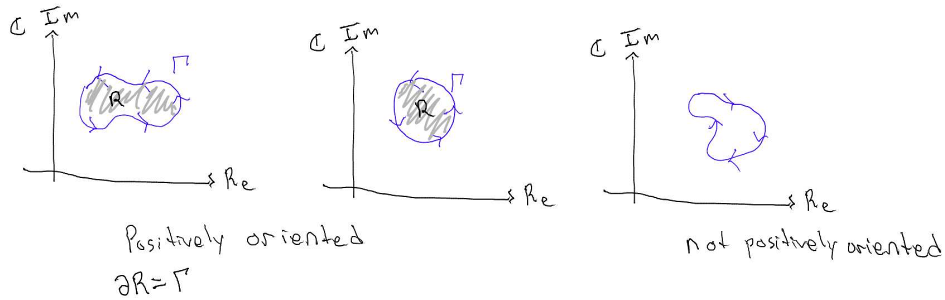

Figure 11.3: Positively oriented simple closed contours in $\mathbb{C}$. A simple closed contour with positive orientation can be thought of as the boundary curve of the region $R \subset \mathbb{C}$ that the contour encloses.

11.1.1 Cauchy-Goursat Theorem

The first useful result that we will discuss regarding integration along closed contours is the *Cauchy-Goursat theorem*.

Theorem 11.1 (Cauchy-Goursat). *Let $f : D \to \mathbb{C}$ be a mapping that is differentiable everywhere on a simply connected domain D. For every closed contour Γ in D, it holds that*

$$\oint_\Gamma f(z)\,dz = 0.$$

Proof. For simplicity, we'll suppose that Γ is a simple closed contour with parameterization $\gamma : [a, b] \to \mathbb{C}$ and positively oriented. Since Γ is simple and closed, we can think of this as being the boundary curve of the region R that it encloses with $\partial R = \Gamma$ (see Figure 11.3). The trick will be to use Green's Theorem to relate the integral along the boundary to an area integral over the region inside. The real and imaginary parts of the points on the path $\gamma(t)$ can be written as functions of t such that $\gamma(t) = x(t) + jy(t)$. The derivative of the path is therefore given by $\gamma'(t) = x'(t) + jy(t)$. We may also separate out $f(x + jy)$ into real and imaginary parts with functions u and v

$$f(x + jy) = u(x, y) + jv(x, y)$$

such that

$$f(\gamma(t)) = u\big(x(t), y(t)\big) + jv\big(x(t), y(t)\big).$$

The contour integral is therefore

$$\begin{aligned}\oint_\Gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \Big(u\big(x(t), y(t)\big) + jv\big(x(t), y(t)\big)\Big)\big(x'(t) + jy'(t)\big)\,dt \\ &= \int_a^b \Big(u\big(x(t), y(t)\big)x'(t) - v\big(x(t), y(t)\big)y'(t)\Big)\,dt + j\int_a^b \Big(v\big(x(t), y(t)\big)x'(t) + u\big(x(t), y(t)\big)y'(t)\Big)\,dt.\end{aligned} \tag{11.1}$$

To continue the computation, notice that we can write the integral on the right side of (11.1) as the sum of two *line integrals* over some *vector fields* where we identify the path $\gamma(t) = x(t) + jy(t)$ in the complex plane with the vector-valued path in $\mathbb{R}^2$ with coordinates

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \text{and derivative} \qquad \boldsymbol{\gamma}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

We simplify the expression in (11.1) to

$$\begin{aligned} \oint_\Gamma f(z)\,dz &= \int_a^b \begin{pmatrix} u\big(x(t),y(t)\big) \\ -v\big(x(t),y(t)\big) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt + j \int_a^b \begin{pmatrix} v\big(x(t),y(t)\big) \\ u\big(x(t),y(t)\big) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt \\ &= \int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,dt + j \int_a^b \boldsymbol{G}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,dt \\ &= \oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} + j \oint_\Gamma \boldsymbol{G} \cdot d\boldsymbol{r} \end{aligned} \tag{11.2}$$

where we define the vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$ in $\mathbb{R}^2$ as

$$\boldsymbol{F}(x,y) = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{G}(x,y) = \begin{pmatrix} v(x,y) \\ -u(x,y) \end{pmatrix}.$$

We have therefore converted a complex contour integral into the sum of path integrals over real vector fields, each of which we may simplify using Green's Theorem as

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \qquad \text{and} \qquad \oint_\Gamma \boldsymbol{G} \cdot d\boldsymbol{r} = \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) dA. \tag{11.3}$$

The partial derivatives of the components of $\boldsymbol{F}$ and $\boldsymbol{G}$ are

$$\frac{\partial F_1}{\partial y} = u_y, \qquad \frac{\partial F_2}{\partial x} = -v_x, \qquad \frac{\partial G_1}{\partial y} = v_y, \qquad \text{and} \qquad \frac{\partial G_2}{\partial x} = u_x.$$

However, since the mapping f is differentiable everywhere in D, the functions u and v satisfy the Cauchy-Riemann equations

$$u_x + v_y = 0 \qquad \text{and} \qquad v_x - u_y = 0,$$

and thus both integrals in (11.3) are zero!

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = 0 \qquad \text{and} \qquad \oint_\Gamma \boldsymbol{G} \cdot d\boldsymbol{r} = 0.$$

It follows from (11.2) that $\oint_\Gamma f(z)\,dz = 0$. □

The key part of the proof of the Cauchy-Goursat Theorem is that we converted the complex contour integral in $\mathbb{C}$ into the sum of two path integrals over real vector fields in $\mathbb{R}^2$,

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} + j \oint_\Gamma \boldsymbol{G} \cdot d\boldsymbol{r},$$

and each of these integrals evaluates to zero as long as f is differentiable everywhere inside of the region R whose boundary is Γ. Indeed, since f is differentiable, these two vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$ are C^1 everywhere in the simply connected region D. These vector fields are also irrotational, since they have zero vorticity

$$\operatorname{vor}(\boldsymbol{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -v_x - u_y = 0 \qquad \text{and} \qquad \operatorname{vor}(\boldsymbol{G}) = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = u_x - v_y = 0.$$

A theorem from vector calculus tells us, since $\boldsymbol{F}$ and $\boldsymbol{G}$ are irrotational on a simply connected domain D, that they must also be conservative. For conservative vector fields, path integrals depend only on the initial and final point of the path (i.e., they are *path independent*). We can expect such an equivalence for complex integrals as well.

Theorem 11.2 (Path independence)**.** *Let $f : D \to \mathbb{C}$ be differentiable everywhere on a simply connected region $D \subseteq \mathbb{C}$. For any two contours Γ_1 and Γ_2 that have the same start and end points, it holds that*

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

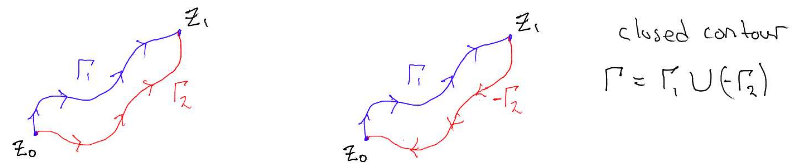

Figure 11.4: Two curves with same start and end point. Reversing Γ_2 and combining the two contours yields a closed contour Γ.

Proof. Let $\Gamma = \Gamma_1 \cup (-\Gamma_2)$ be the contour comprising the contour Γ_1 followed by the contour Γ_2 in reverse. Then Γ is a closed contour in a simply connected region D. Since f is differentiable everywhere in D, the Cauchy-Goursat theorem tells us that

$$\int_{\Gamma} f(z)\,dz = 0.$$

However, we may also compute this integral by splitting it up into its two parts as

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz,$$

and thus

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

□

If $f : D \to \mathbb{C}$ is differentiable everywhere in a simply connected region, we can unambiguously interpret the value of the integral

$$\int_{z_0}^{z_1} f(z)\,dz$$

where we take any contour that starts at z_0 and ends at z_1.

Example 11.3. Recall the example from Week 10 where we integrated the mapping $f(z) = z^2$ along two different contours: the contour Γ_1 which was the straight line segment from 0 to $1 + j$, and the contour Γ_2 consisting of the straight line segment from 0 to 1 followed by the straight line segment from 1 to $1 + j$. Both contours start at 0 and end at $1 + j$. Since the mapping $f(z) = z^2$ is differentiable everywhere with derivative $f'(z) = 2z$, the values of the integrals along Γ_1 and Γ_2 will be the same.

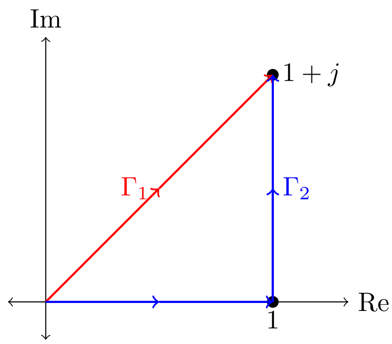

Figure 11.5: Two contours connecting 0 and $1 + j$.

11.1.2 Deformation of paths

It is useful to think of path independence as a process of *path deformation*. That is, we can imagine continuously deforming Γ_1 into Γ_2 while keeping the endpoints fixed. If f is differentiable in the region and we don't have to cross any points where f is not differentiable when deforming Γ_1 into Γ_2, then

$$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz.$$

This idea of path deformation can be applied to closed paths as well.

Let Γ_1 and Γ_2 be two simple closed contours with the same orientation that do not cross each other, and suppose f is a mapping that is differentiable on Γ_1, Γ_2, and on the entire region contained between the two curves (see Figure 11.6).

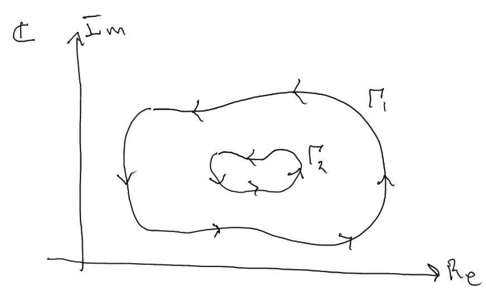

Figure 11.6: Two non-intersecting simple closed contours with the same orientation. If a mapping f is differentiable everywhere in between Γ_1 and Γ_2, we can 'deform' one contour into the other.

It is possible that f is not differentiable inside the inner contour, so we may not use the Cauchy-Goursat theorem to say that these integrals are both zero. However, we may introduce few 'slits' into the contours to create simply connected regions where f is differentiable. We'll slit the region between Γ_1 and Γ_2 by adding two contours Γ_3 and Γ_4 connecting the two contours (see Figure 11.7), which splits each of the contours Γ_1 and Γ_2 into two contours Γ_1' and Γ_1'' and Γ_2 and Γ_2''. This splits the region between Γ_1 and Γ_2 into two separate regions, D_1 and D_2, each of which is simply connected, and the boundaries of these regions consist of the composite contours

$$\partial D_1 = \Gamma_1' \cup \Gamma_3 \cup (-\Gamma_2') \cup (-\Gamma_4) \qquad \text{and} \qquad \partial D_2 = \Gamma_1'' \cup \Gamma_4 \cup (-\Gamma_2'') \cup (-\Gamma_3).$$

Since the regions D_1 and D_2 are simply connected and f is differentiable everywhere on D_1 and D_2, we

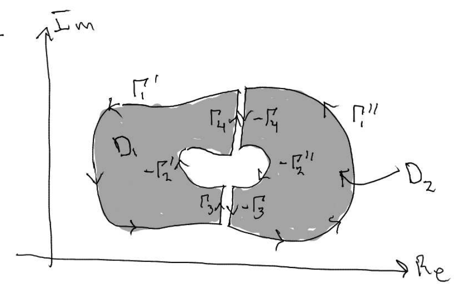

Figure 11.7: Adding a few 'slices' to the countours allows us to split the non-simply connected region into two simply connected regions.

have that

$$\oint_{\partial D_1} f(z)\,dz = 0 \qquad \text{and} \qquad \oint_{\partial D_2} f(z)\,dz = 0.$$

Summing together each of the pieces of the contour integrals along ∂D_1 and ∂D_2, we see that the integrals along Γ_3 and Γ_4 cancel out the integrals along $-\Gamma_3$ and $-\Gamma_4$, and we get

$$\begin{aligned} 0 &= \oint_{\partial D_1} f(z)\,dz + \oint_{\partial D_2} f(z)\,dz \\ &= \oint_{\partial\Gamma_1'} f(z)\,dz + \oint_{\Gamma_1''} f(z)\,dz - \oint_{\partial\Gamma_2'} f(z)\,dz - \oint_{\Gamma_2''} f(z)\,dz \\ &= \oint_{\Gamma_1} f(z)\,dz - \oint_{\partial\Gamma_2} f(z)\,dz. \end{aligned}$$

It follows that

$$\oint_{\Gamma_1} f(z)\,dz = \oint_{\partial\Gamma_2} f(z)\,dz.$$

This is called the *principle of deformation of paths*, which we describe as follows. If a contour Γ_1 can be continuously deformed into another contour Γ_2 without pulling the contour over any points where the mapping f is not differentiable, then the integrals of f around Γ_1 and Γ_2 are the same.

Theorem 11.4 (Principle of deformation of paths)**.** *Let Γ_1 and Γ_2 be two closed simple positively oriented contours in $\mathbb{C}$. If a mapping f is differentiable everywhere on the region on and in between the contours Γ_1 and Γ_2, then*

$$\oint_{\Gamma_1} f(z)\,dz = \oint_{\Gamma_2} f(z)\,dz.$$

Example 11.5. Let Γ be any simple closed contour that goes counterclockwise around the origin. Evaluate

$$\oint_{\Gamma} \frac{1}{z}\,dz.$$

Solution. We have already computed this integral in the case when Γ is the unit circle centred at the origin. Since the mapping $f(z) = 1/z$ is differentiable everywhere except at the origin $z = 0$, we can 'deform' our contour Γ into the unit circle without crossing over the origin, and thus integrating over these two contours will give the same answer. It follows that

$$\oint_{\Gamma} \frac{1}{z}\,dz = 2\pi j$$

for *any* simple closed contour that goes counterclockwise around the origin.

From Example 11.5, we see that integrating a mapping f around a closed contours depends only on the points inside the contours where f is not differentiable. If f is differentiable everywhere inside Γ, then

$$\oint_\Gamma f(z)\,dz = 0$$

by Cauchy-Goursat. Otherwise, we may have small contributions to the integral that come from points where the mapping is not differentiable, such as the point $z = 0$ in the mapping $f(z) = 1/z$. In fact, most of the complex mappings we will deal with in this course will be differentiable *almost* everywhere except at some small number of separate points. Points where a mapping f is either not defined or not differentiable are called *singularities*. Computing a complex integral of a mapping f over a closed contour Γ will amount to identifying all of the singularities of f that lie inside the contour Γ.

11.1.3 An important integral

The idea from Example 11.5 can be extended to other mappings of the form

$$f(z) = \frac{1}{z - z_0}$$

for points $z_0 \in CC$. Let Γ_1 and Γ_2 be positively oriented simple closed contours that goes around z_0. By the principle of deformation of paths, the values of the integrals

$$\oint_{\Gamma_1} \frac{1}{z - z_0} \qquad \text{and} \qquad \oint_{\Gamma_2} \frac{1}{z - z_0}$$

will be equal, since we can deform one into the other without crossing over the only point $z = z_0$ where the mapping $f(z) = 1/(z - z_0)$ is not differentiable. Hence we might as well choose a contour that yields the simplest integral. Let Γ_a be the circle of radius a centered at z_0 oriented counterclockwise, which can be parameterized as

$$\gamma(t) = z_0 + ae^{jt}$$

for $t \in [0, 2\pi]$. This path has derivative $\gamma'(t) = aje^{jt}$, and thus

$$\begin{aligned}\oint_{\Gamma_a} \frac{1}{z - z_0}\,dz &= \int_0^{2\pi} \frac{1}{(z_0 + ae^{jt}) - z_0} aje^{jt}\,dt \\ &= \int_0^{2\pi} \frac{1}{2\pi} \frac{1}{ae^{jt}} aje^{jt}\,dt = j\int_0^{2\pi} dt = 2\pi j.\end{aligned}$$

This is independent of the choice of radius a (as we should expect from the principle of path deformation, since it should be the same for any path that goes counterclockwise around $z = z_0$). We therefore obtain the following result: For *any* simple closed contour that goes counterclockwise around a point z_0, it holds that

$$\oint_\Gamma (z - z_0)^{-1}\,dz = \oint_\Gamma \frac{1}{z - z_0}\,dz = 2\pi j.$$

What happens if we change the integral to

$$\oint_\Gamma (z - z_0)^n\,dz$$

for integers n other than $n = -1$? There are a few different cases to consider.

(i) If $n \geq 0$ then the mapping $f(z) = (z - z_0)^n$ is differentiable everywhere, and thus

$$\oint_\Gamma (z - z_0)^n \, dz = 0$$

for all closed contours by the Cauchy-Goursat theorem.

(ii) If $n = -1$, the above result tells us that

$$\oint_\Gamma (z - z_0)^{-1} \, dz = 2\pi j$$

for any simple closed contour going counterclockwise around z_0.

(iii) If $n < -1$, the mapping $f(z) = (z - z_0)^n$ is differentiable everywhere except at $z = z_0$, so we may use the principle of path deformation to deform the path into a circle of radius a centered at $z = z_0$. As before, we may integrate this directly by parameterizing the contour as $\gamma(t) = z_0 + ae^{jt}$ and computing

$$\begin{aligned}\oint_{\Gamma_a} (z - z_0)^n \, dz &= \int_0^{2\pi} \left((z_0 + ae^{jt}) - z_0\right)^n aje^{jt} \, dt \\ &= aj \int_0^{2\pi} \left(ae^{jt}\right)^n e^{jt} \, dt \\ &= a^{n+1} j \int_0^{2\pi} e^{jt(n+1)} \, dt \\ &= \frac{a^{n+1}}{n+1} \underbrace{e^{jt(n+1)}\Big|_0^{2\pi}}_{=1-1=0} = 0,\end{aligned}$$

where we note that $n + 1 \neq 0$ since $n < -1$.

Combining all of these results together, we obtain the following important family of integrals: Suppose Γ is any simple closed positively oriented contour that goes around a point z_0. For all integers $n \in \mathbb{Z}$, we have

$$\boxed{\oint_\Gamma (z - z_0)^n \, dz = \begin{cases} 2\pi j & n = -1 \\ 0 & n \neq -1. \end{cases}} \tag{11.4}$$

Example 11.6. Evaluate the integral

$$\oint_\Gamma \frac{1}{z^2(z-2)(z-4)} \, dz$$

counterclockwise around the contour Γ shown below.

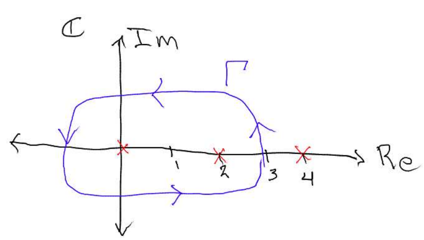

Figure 11.8: The contour for the integral in Example 11.6.

Solution. The mapping $f(z) = \frac{1}{z^2(z-2)(z-4)}$ is differentiable everywhere except at its singularities at $z = 0$, $z = 2$, and $z = 4$. However, only the singularities at $z = 0$ and $z = 2$ are contained inside of the contour Γ. We may use the method of partial fractions to factor the integrand as

$$\frac{1}{z^2(z-2)(z-4)} = \frac{3}{32}\frac{1}{z} + \frac{1}{8}\frac{1}{z^2} - \frac{1}{8}\frac{1}{z-2} + \frac{1}{32}\frac{1}{z-4}$$

and integrate each term separately. Now,

$$\oint_\Gamma \frac{1}{z}\,dz = 2\pi j \qquad \text{(by (11.4), where } z_0 = 0 \text{ and } n = -1)$$

$$\oint_\Gamma \frac{1}{z^2}\,dz = 0 \qquad \text{(by (11.4), where } z_0 = 0 \text{ and } n = -2)$$

$$\oint_\Gamma \frac{1}{z-2}\,dz = 2\pi j \qquad \text{(by (11.4), where } z_0 = 2 \text{ and } n = -1)$$

$$\oint_\Gamma \frac{1}{z-4}\,dz = 0 \qquad \text{(since the singularity } z = 4 \text{ is not inside the contour } \Gamma)$$

The integral can be evaluated as

$$\begin{aligned}\oint_\Gamma \frac{1}{z^2(z-2)(z-4)}\,dz &= \frac{3}{32}\underbrace{\oint_\Gamma \frac{1}{z}\,dz}_{=2\pi j} + \frac{1}{8}\underbrace{\oint_\Gamma \frac{1}{z^2}\,dz}_{=0} - \frac{1}{8}\underbrace{\oint_\Gamma \frac{1}{z-2}\,dz}_{=2\pi j} + \frac{1}{32}\underbrace{\oint_\Gamma \frac{1}{z-4}\,dz}_{=0} \\ &= \frac{3}{32}(2\pi j) + \frac{1}{8}(0) - \frac{1}{8}(2\pi j) + \frac{1}{32}(0) \\ &= -\frac{j\pi}{16}.\end{aligned}$$

11.1.4 Fundamental theorem of complex integration

A final rule of complex integration that we will consider is the so-called *fundamental theorem of complex integral calculus*. Before obtaining this result, let us recall the *fundamental theorem of calculus*. One version of this theorem is as foll lows: Let $f : [x_0, x_1] \to \mathbb{R}$ be a function that is continuously differentiable on the interval $[x_0, x_1]$, then

$$\int_{x_0}^{x_1} f'(x)\,dx = f(x_1) - f(x_0).$$

There is also a *fundamental theorem of line integrals* that we discussed previously in the course. For functions in $\mathbb{R}^2$, this can be stated as: Let $f : D \to \mathbb{R}^2$ be C^1 in a connected region $D \subseteq \mathbb{R}^2$ and let Γ be a curve in D that starts at $\boldsymbol{r}_0$ and ends at $\boldsymbol{r}_1$, then

$$\int_\Gamma \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r_1}) - f(\boldsymbol{r_0}).$$

In particular, if $\boldsymbol{F}$ is a conservative field (i.e., $\boldsymbol{F} = \nabla f$ for some C^1 scalar field f), then line integrals are *independent of path* since they depend only on the start and endpoints of the curve and we can unambiguously interpret the integral

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \nabla f \cdot d\boldsymbol{r}.$$

We have a similar important result for complex integration. To derive this, we use a similar strategy in the proof of the Cauchy-Goursat theorem. namely, we interpret the contour Γ in $\mathbb{C}$ with parameterization $\gamma : [a, b] \to \mathbb{C}$ as a curve in $\mathbb{R}^2$ with parameterization $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$.

Theorem 11.7 (Fundamental theorem of complex integration)**.** *Let $f : D \to \mathbb{C}$ be a mapping that is differentiable everywhere in a connected region $D \subseteq \mathbb{C}$. For any contour Γ in D, it holds that*

$$\int_\Gamma f'(z)\,dz = f(z_1) - f(z_0), \tag{11.5}$$

where z_0 and z_1 are the start and end points of the contour.

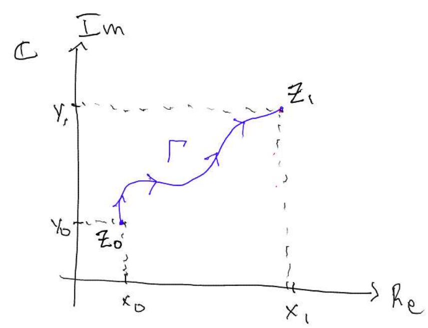

Figure 11.9: A contour that starts at $z_0 = x_0 + jy_0$ and ends at $z_1 = x_1 + jy_1$.

Proof. Suppose that $f(z)$ can be decomposed in Cartesian form as

$$f(x + jy) = u(x, y) + jv(x, y)$$

for some functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$. Let $\gamma : [a, b] \to \mathbb{C}$ be a parameterization of Γ with components

$$\gamma(t) = x(t) + jy(t),$$

and let z_0 and z_1 have components $z_0 = x_0 + jy_0$ and $z_1 = x_1 + jy_1$. The desired integral can be computed as

$$
\begin{aligned}
\int_\Gamma f'(z)\,dz &= \int_a^b f'(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b \Big(u_x\big(x(t),y(t)\big) + jv_x\big(x(t),y(t)\big)\Big)\big(x'(t)+jy'(t)\big)\,dt \\
&= \int_a^b \Big(u_x\big(x(t),y(t)\big)x'(t) - v_x\big(x(t),y(t)\big)y'(t)\Big)\,dt \\
&\qquad\qquad + j\int_a^b \Big(v_x\big(x(t),y(t)\big)x'(t) + u_x\big(x(t),y(t)\big)y'(t)\Big)\,dt \qquad (11.6)\\
&= \int_a^b \Big(u_x\big(x(t),y(t)\big)x'(t) + u_y\big(x(t),y(t)\big)y'(t)\Big)\,dt \\
&\qquad\qquad + j\int_a^b \Big(v_x\big(x(t),y(t)\big)x'(t) + v_y\big(x(t),y(t)\big)y'(t)\Big)\,dt \qquad (11.7)\\
&= \int_a^b \nabla u(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt + j\int_a^b \nabla v(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt \qquad (11.8)\\
&= u(\boldsymbol{\gamma}(b)) - u(\boldsymbol{\gamma}(a)) + j\big(v(\boldsymbol{\gamma}(b)) - v(\boldsymbol{\gamma}(a))\big) \qquad (11.9)\\
&= u(x_1,y_1) + jv(x_1,y_1) - \big(u(x_0,y_0) + jv(x_0,y_0)\big) \\
&= f(x_1+jy_1) - f(x_0+jy_0) \\
&= f(z_1) - f(z_1),
\end{aligned}
$$

where, to get from line (11.6) to (11.7) we use the assumption that f is differentiable and thus u and v satisfy the Cauchy-Riemann equations

$$v_x = -u_y \qquad \text{and} \qquad u_x = v_y,$$

and in (11.8) we view the path $\gamma : [a,b] \to \mathbb{C}$ with components $\gamma(t) = x(t) + jy(t)$ as a path in $\mathbb{R}^2$ with components $\boldsymbol{\gamma}(t) = \big(x(t),\, y(t)\big)$ such that we can convert the integral to a line integral of the gradients of u and v. Since the line integral of a conservative vector field depends only on the initial and end points, we obtain the desired result. □

Example 11.8. Consider again our example

$$\int_\Gamma z^2\,dz$$

where Γ was any path that starts at 0 and ends at $1+j$. The mapping $f(z) = z^2$ is differentiable everywhere. By the fundamental theorem of complex integral we have,

$$\int_\Gamma z^2\,dz = \int_0^{1+j} z^2\,dz = \frac{z^3}{3}\bigg|_0^{1+j} = \frac{(1+j)^3}{3} - 0 = \frac{2(j-1)}{3}.$$

11.2 Cauchy's integral formula

Recall that the Cauchy-Goursat theorem depends on the differentiability of the mapping. If f has any singularities inside of a closed contour we must determine the value of the integral around each singularity. Here we will show how to integrate over certain types of singularities.

Let $f : D \to \mathbb{C}$ be a differentiable mapping in a simply connected region $D \subset \mathbb{C}$, let Γ be any positively oriented simple closed contour in D that goes around some fixed point $z_0 \in D$, and consider the integral

$$\oint_\Gamma \frac{f(z)}{z - z_0}\,dz \qquad (11.10)$$

(see Figure 11.10).

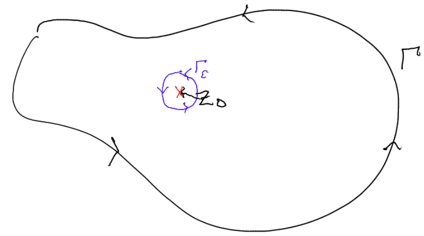

Figure 11.10: A positively oriented simple closed contour Γ going around a point z_0. We can deform the contour to a small circle Γ_ε centred at the singularity at $z = z_0$.

To evaluate the integral in (11.10), we begin by making use of the Principal of Deformation of Paths to deform the contour into a circular contour Γ_ε of radius ε centred at z_0 with ε small enough so that Γ_ε lies entirely inside Γ, as shown in Figure 11.10. This deformation is justified since the integrand $f(z)/(z - z_0)$ is differentiable everywhere between Γ and Γ_ε.

Note that the integral of $f(z)/(z - z_0)$ around Γ_ε is independent of the choice of radius ε, provided that Γ_ε stays entirely within Γ. Thus we can take the limit as $\varepsilon \to 0$. This is convenient since, for small enough values of ε, we may approximate $f(z) \approx f(z0)$ for z on the contour Γ_ε. As the contour Γ_ε can be parameterized as $\gamma(t) = z_0 + \varepsilon e^{jt}$ for $t \in [0, 2\pi]$, we have

$$\begin{aligned}
\oint_\Gamma \frac{f(z)}{z - z_0}\,dz = \lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} \frac{f(z)}{z - z_0}\,dz &= \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{jt})}{(z_0 + \varepsilon e^{jt}) - z_0} j\varepsilon e^{jt}\,dt \\
&= \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{jt})}{\varepsilon e^{jt}} j\varepsilon e^{jt}\,dt \\
&= j \lim_{\varepsilon\to 0} \int_0^{2\pi} f(z_0 + \varepsilon e^{jt})\,dt \\
&= j \int_0^{2\pi} \lim_{\varepsilon\to 0} f(z_0 + \varepsilon e^{jt})\,dt \\
&= j \int_0^{2\pi} f(z_0)\,dt = 2\pi j f(z_0).
\end{aligned}$$

This result is known as *Cauchy's integral formula.*

Theorem 11.9 (Cauchy's integral formula)**.** *Let $f : D \to \mathbb{C}$ be differentiable in a simply connected region D and let Γ be a positively oriented simple closed curve going around a point z_0 in D. Then*

$$\oint_\Gamma \frac{f(z)}{z - z_0}\,dz = 2\pi j f(z_0). \tag{11.11}$$

Example 11.10. Evaluate the integral

$$\oint_\Gamma \frac{\cos z}{z - 1}\,dz$$

where Γ is the circle of radius 2 centred at the origin oriented counterclockwise.

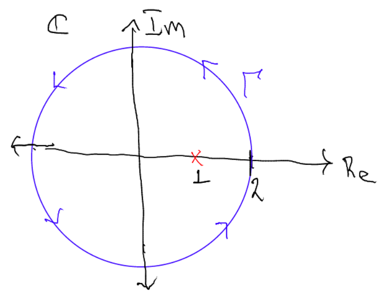

Figure 11.11: The contour from Example 11.10.

Solution. First note from Figure 11.11 that the singularity $z = 1$ is located inside of the contour. Setting $f(z) = \cos z$, we may use Cauchy's Integral Formula to find that

$$\oint_\Gamma \frac{\cos z}{z-1}\, dz = 2\pi j \cos 1.$$

Note that if the singularity at $z = 1$ was not contained inside the contour Γ then the resulting integral would be zero since the integrand would be differentiable everywhere inside the contour.

Example 11.11. Evaluate the integral

$$\oint_\Gamma \frac{z}{(9-z^2)(z+j)}\, dz$$

where Γ is the circle of radius 2 centred at the origin oriented counterclockwise.

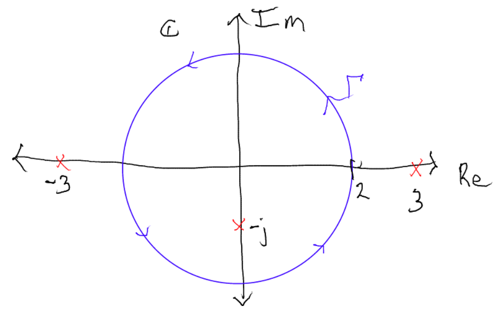

Figure 11.12: The contour from Example 11.11.

Solution. Here, we see that the integrand has singularities as $z = \pm 3$ and at $z = -j$. From 11.12, we see that only the singularity at $z = -j$ is located inside the contour. If we choose the function $f(z) = z/(9 - z^2)$, which is differentiable everywhere inside the contour Γ, we may use Cauchy's Integral Formula to find that

$$\oint_\Gamma \frac{z}{(9-z^2)(z+j)}\, dz = \oint_\Gamma \frac{f(z)}{z+j}\, dz = 2\pi j f(-j) = 2\pi j \frac{-j}{9-(-j)^2} = \frac{2\pi}{9+1} = \frac{\pi}{5}.$$

The Cauchy integral formula allows us to integrate functions that have "first order singularities" of the form $f(z)/(z - z_0)$ for functions f that are differentiable at $z = z_0$. It does not apply if the singularity is of higher order, that is if the integrand is of the form $f(z)/(z - z_0)^2$, $f(z)/(z - z_0)^3$, or so on. To deal with these cases, we may start by using using the limit definition to compute the derivative of f and make use of the Cauchy integral formula. That is, at $z = z_0$ the derivative of f is given by

$$f'(z_0) = \lim_{\Delta\to 0} \frac{f(z_0 + \Delta_z) - f(z_0)}{\Delta z},$$

which we may evaluate by choosing a simple closed contour Γ small enough so that f is differentiable everywhere inside Γ and use the Cauchy integral formula to find that

$$f(z_0) = \frac{1}{2\pi j}\oint_\Gamma \frac{f(z)}{z-z_0}\,dz \qquad \text{and} \qquad f(z_0) = \frac{1}{2\pi j}\oint_\Gamma \frac{f(z)}{z-z_0-\Delta z}\,dz.$$

The derivative of f at z_0 can therefore be determined by

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta_z)-f(z_0)}{\Delta z} \\
&= \frac{1}{2\pi j}\lim_{\Delta z\to 0}\left(\frac{1}{\Delta z}\left(\oint_\Gamma \frac{f(z)}{z-z_0-\Delta z}\,dz - \oint_\Gamma \frac{f(z)}{z-z_0}\,dz\right)\right) \\
&= \frac{1}{2\pi j}\lim_{\Delta z\to 0}\left(\frac{1}{\Delta z}\oint_\Gamma \left(\frac{f(z)}{z-z_0-\Delta z} - \frac{f(z)}{z-z_0}\right)dz\right) \\
&= \frac{1}{2\pi j}\lim_{\Delta z\to 0}\left(\frac{1}{\Delta z}\oint_\Gamma \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)}\,dz\right) \\
&= \frac{1}{2\pi j}\lim_{\Delta z\to 0}\oint_\Gamma \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz \\
&= \frac{1}{2\pi j}\oint_\Gamma \frac{f(z)}{(z-z_0)^2}\,dz,
\end{aligned}$$

which yields the formula

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi j f'(z_0).$$

for any positively oriented simple closed contour Γ that goes around a point z_0 in a simply connected region D (and assuming that f is differentiable everywhere in D).

Similarly, we can repeat this process of differentiation as many times as we would like to obtain the formula

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi j}{n!}f^{(n)}(z_0)$$

for any nonnegative integer $n \geq 0$, where $f^{(n)}$ indicates the n^{th} order derivative of f (and $f^{(0)} = f$). This result is known as the *generalized Cauchy integral formula.*

Theorem 11.12 (Generalized Cauchy integral formula)**.** *Let $f : D \to \mathbb{C}$ be differentiable in a simply connected region D and let Γ be a positively oriented simple closed curve going around a point z_0 in D. For any integer $n \geq 0$,*

$$n!\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi j f^{(n)}(z_0). \tag{11.12}$$

Example 11.13. Evaluate the integral

$$\oint_\Gamma \frac{e^{2z}}{z^4}\,dz$$

where Γ is the circle of radius 1 centred at the origin oriented counterclockwise.

Solution. Note that the singularity at $z = 0$ is inside the contour. Setting $f(z) = e^{2z}$, we may rewrite the integral as

$$\oint_\Gamma \frac{f(z)}{(z-0)^{3+1}}\,dz$$

and use the generalized Cauchy integral formula. Note that $f^{(3)}(z) = 8e^{2z}$ and thus

$$\oint_\Gamma \frac{f(z)}{(z-0)^{3+1}}\,dz = \frac{2\pi j}{3!} f^{(3)}(0) = \frac{8\pi}{3} j.$$

Example 11.14. Evaluate the integral from Example 11.6

$$\oint_\Gamma \frac{1}{z^2(z-2)(z-4)}\,dz$$

but this time using the generalized Cauchy integral formula.

Solution. Note that the contour Γ only contains the singularities $z = 0$ and $z = 2$. We may use the principle of deformation of paths to deform the contour Γ into the the contour shown in Figure 11.13, which consists of two small circles Γ_0 and Γ_2 centered at $z = 0$ and $z = 2$ respectively and the line along the real axis that connects them (where this line segment is traversed twice: once backwards and once forwards). Hence

$$\oint_\Gamma \frac{1}{z^2(z-2)(z-4)}\,dz = \oint_{\Gamma_0} \frac{1}{z^2(z-2)(z-4)}\,dz + \oint_{\Gamma_2} \frac{1}{z^2(z-2)(z-4)}\,dz,$$

since the integrals along the line segment on the real axis cancel each other out.

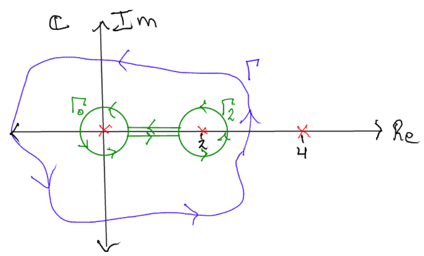

Figure 11.13: Deforming the contour Γ into two smaller contours Γ_0 and Γ_2 around the two singularities $z = 0$ and $z = 2$.

To compute each of the separate integrals around the singularities $z = 0$ and $z = 2$, we can express the integrand two different ways:

$$\frac{1}{z^2(z-2)(z-4)} = \frac{f(z)}{z^2} \qquad \text{where} \qquad f(z) = \frac{1}{(z-2)(z-4)} = \frac{1}{z^2-6z+8}$$

and

$$\frac{1}{z^2(z-2)(z-4)} = \frac{g(z)}{z-2} \qquad \text{where} \qquad g(z) = \frac{1}{z^2(z-4)} = \frac{1}{z^3-4z^2}.$$

We may then use the generalized Cauchy integral formula to compute each piece as

$$\oint_{\Gamma_0} \frac{1}{z^2(z-2)(z-4)}\,dz = \oint_{\Gamma_0} \frac{f(z)}{z^2}\,dz = 2\pi j f'(0) = \frac{3\pi}{16} j,$$

where we note that

$$f'(z) = -\frac{2z-6}{(z^2-6z+8)^2} \qquad \text{and thus} \qquad f'(0) = \frac{6}{8^2} = \frac{3}{32},$$

and

$$\oint_{\Gamma_2} \frac{1}{z^2(z-2)(z-4)}\,dz = \oint_{\Gamma_2} \frac{g(z)}{z-2}\,dz = 2\pi j g(2) = -\frac{\pi}{4} j.$$

The desired integral is therefore

$$\oint_\Gamma \frac{1}{z^2(z-2)(z-4)}\,dz = \frac{3\pi}{16} j - \frac{\pi}{4} j = -\frac{\pi}{16} j.$$

Remark 11.15. From the previous example, we see that integrating a mapping f around a closed contour Γ depends only on the singularities of f that lie inside the contour Γ. Summing up the values of the contour around each singularity separately gives us the final answer.