ECE 206 – University of Waterloo **Fall 2019**

Lecture notes for Week 12

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12.1 Taylor series and Laurent series

12.1.1 Analytic functions

So far we have developed methods for complex integration, concluding last week with Cauchy's integral formula which we will review here. Suppose f is differentiable everywhere inside of a simple connected region $D \subseteq \mathbb{C}$ and let Γ be a positively oriented simple closed curve going around some point $z_0 \in D$. For any integer $n \geq 0$, Cauchy's integral formula states that

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \tag{12.1}$$

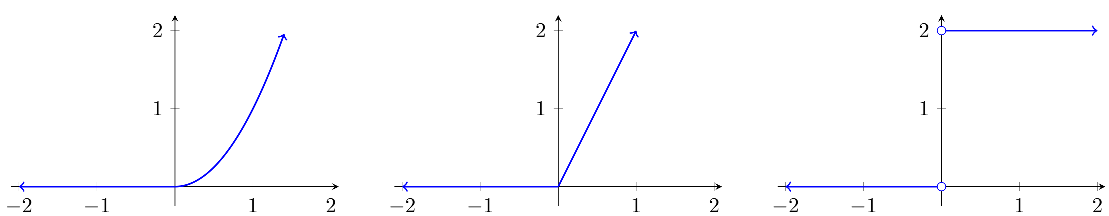
where $f^{(n)}$ is the n^{th} derivative of f (and $f^{(0)} = f$). Although we did not mention this before, this is actually quite a remarkable fact. We have only assumed that f was once differentiable on D, but Cauchy's integral formula tells us that f is actually *infinitely* differentiable at z_0 (where the value of all of the derivatives can be found from the formula in (12.1))! This is a special property of complex-differentiable functions that is certainly *not* true for regular functions of *real* variables. As an example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0. \end{cases} \tag{12.2}$$

This function is differentiable everywhere with derivative given by

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0, \end{cases} \tag{12.3}$$

but f' is certainly not differentiable at $x = 0$ (since it has a 'kink' there). Differentiability of complex functions is different. Our discussion above tells us that if a complex-valued function is *complex-differentiable* everywhere, then it must also be infinitely differentiable everywhere!



(a) Plot of f in (12.2). (b) Plot of f' in (12.3). (c) Partial plot of f''.

Figure 12.1: An example of a function of real variables that is once differentiable but not twice differentiable.

This sort of property has a special name that is common in the parlance of complex analysis but we have so far not used. Before continuing, we must first define the concept of a *disk*. Let $R > 0$ be a positive real number. The *disk* of radius R centred at a point z_0 is the set $D \subseteq \mathbb{C}$ defined by

$$D = \{z \in \mathbb{C} \,|\, |z - z_0| < R\}.$$

Intuitively, it consists of all points whose distance away from z_0 is strictly less than R. (See Figure 12.2.)

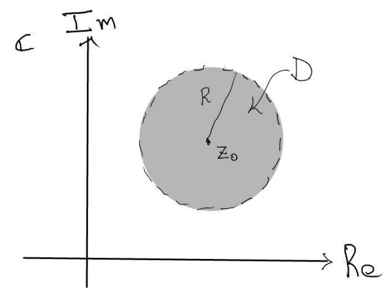


Figure 12.2: The disk D of radius R centred at z_0.

> **Definition 12.1.** Let f be a mapping and z_0 a point in the complex plane. We say that f is *analytic at a point* z_0 if there exists a disk D_R with positive radius $R > 0$ such that f is differentiable everywhere inside D_R.

From our discussion above about Cauchy's integral formula in (12.1), we see that if a function is analytic at z_0 then it is infinitely differentiable. So we may think of a function being "analytic" in some region D as the same as that function being "infinitely differentiable" in that region.

We should remark that the concept of "complex-differentiable" is distinct from "analytic." A function might be differentiable at a point but not anywhere nearby to that point, such as the function $f(z) = |z|$ which is only differentiable at the point $z = 0$ and nowhere else. However, this function is not *analytic* at $z = 0$, since analycity requires that f also be differentiable in some small disk around the point $z = 0$. From now on, we will use the term "analytic" rather than "differentiable" to when discussing differentiability. This will be useful when we begin to discuss Taylor series in the next section.

12.1.2 Taylor series

You are hopefully familiar with the concept of *Taylor series* from your previous calculus courses. We will first recall a few basic definitions and facts before discussing Taylor series of complex functions.

> **Definition 12.2.** A *power series* around a point $z_0 \in \mathbb{C}$ is an infinite sum of the form
>
> $$\sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$
>
> for some coefficients $c_0, c_1, c_2, \cdots \in \mathbb{C}$. The power series is said to *converge* for a value $z \in \mathbb{C}$ if the limit
>
> $$\lim_{N\to\infty} \sum_{n=0}^{N} c_n(z-z_0)^n$$
>
> exists. If it does not converge then it is said to *diverge*. The largest value of R such that the power series converges for all z satisfying $|z - z_0| < R$ is called the *radius of convergence*.

We will not be too concerned with techniques for proving whether or not a particular series diverges. Instead, we'll turn directly to our discussion of Taylor series. In Section 12.1.1, we found that if a function is *analytic* at a point z_0 (i.e., differentiable at z_0 and in some small region around z_0), then it admits derivatives of all orders there. Since $f(z_0)$, $f'(z_0)$, $f''(z_0)$, $\cdots$ all exist, it is at least formally possible to write down the *Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{12.4}$$

of $f(z)$ about the point z_0 whether or not it converges. However, whether or not the Taylor series expansion is *useful* or *meaningful* will depend on answers to these questions: Does the series converge and, if so, for which values of z? If it does converge, what does it converge to? Although we will not be concerned with a proof, the following fact tells us the usefulness of Taylor series' of functions and answers some of the above questions.

Theorem 12.3 (Taylor's theorem)**.** *Let f be analytic everywhere on a disk D_R of radius R centred at a point z_0. Then the power series in (12.4) converges and*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{12.5}$$

holds for all z satisfying $|z - z_0| < R$.

The main implication of Taylor's Theorem is that, if f is analytic in some disk $\{z \mid |z - z_0| < R\}$, then $f(z)$ can be represented in that disk by a convergent power series (namely, it's Taylor series about z_0). On the other hand, if R is the radius of convergence of the series in (12.5), then the series (12.5) diverges for all z satisfying $|z - z_0| > R$. If f is analytic everywhere, we simply say that its radius of convergence is $R = \infty$. In Table 12.1 we collect a few familiar Taylor series about $z_0 = 0$.

$f(z)$	Taylor series		Radius of convergence
$e^z =$	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$	$= 1 + z + \frac{z}{2!} + \frac{z^3}{3!} + \cdots$	$R = \infty$
$\sin z =$	$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$	$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$	$R = \infty$
$\cos z =$	$\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$	$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$	$R = \infty$
$\frac{1}{1-z} =$	$\sum_{n=0}^{\infty} z^n$	$= 1 + z + z^2 + z^3 + \cdots$	$R = 1$

Table 12.1: Some familiar Taylor series for complex functions

Remark 12.4. How does one determine the radius of convergence of a Taylor series? Well, if a mapping f is analytic at a point z_0, then there must be some positive radius $R > 0$ such that f is analytic in the disk of radius R centred at z_0 (and thus the Taylor series converges for all points z satisfying $|z - z_0| < R$). However, if f is *not* analytic (or perhaps not even defined) at some other point z_1, then the Taylor series of f about z_0 cannot converge on a disk of radius R if $|z_1 - z_0| < R$. Hence, the radius of convergence of the Taylor series of f about z_0 must be equal to the distance of z_0 to the nearest nearest singularity of f.

This helps to explain why the radius of convergence of $f(z) = 1/(1-z)$ about $z_0 = 0$ is equal to 1. The only singularity of $1/(1-z)$ is at $z = 1$, and the distance from the point 0 to the point 1 is exactly 1. The mapping $f(z) = 1/(1-z)$ is analytic at any other point z satisfying $|z-1| < 1$.

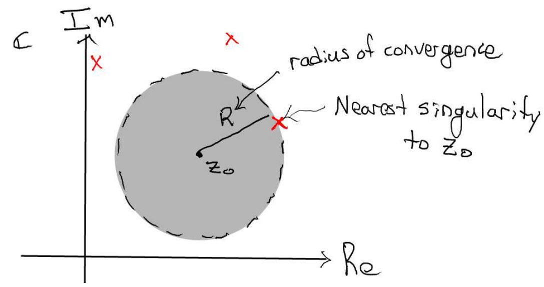


Figure 12.3: If a mapping f is analytic at a point z_0, the radius of convergence of the Taylor series of f about z_0 is equal to the distance from z_0 to the nearest singularity of f.

We can use Taylor series expansions of known mappings to find Taylor series' for other functions, as the following few examples show.

Example 12.5. Find the Taylor series expansions of the following mappings about the given points and determine the radius of convergence.

(i) $f(z) = e^{z^2}$ about $z_0 = 0$

(ii) $f(z) = 1/(z^2+1)$ about $z = 0$

(iii) $f(z) = 1/z$ about $z = 1$

Solution. (i) Here we may simply use the known expansion for e^w where we set $w = z^2$ to get

$$e^{z^2} = e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{3!} + \cdots = 1 + z^2 + \frac{z^4}{2} + \frac{z^6}{3!} + \cdots =$$

which converges everywhere.

(ii) We may use the known Taylor series expansion for $1/(1-w)$ about $w = 0$ where we set $w = -z^2$. That is,

$$\begin{aligned}\frac{1}{z^2+1} = \frac{1}{1-w} &= 1 + w + w^2 + w^3 + \cdots \\ &= 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \cdots \\ &= 1 - z^2 + z^4 - z^6 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n z^{2n}.\end{aligned}$$

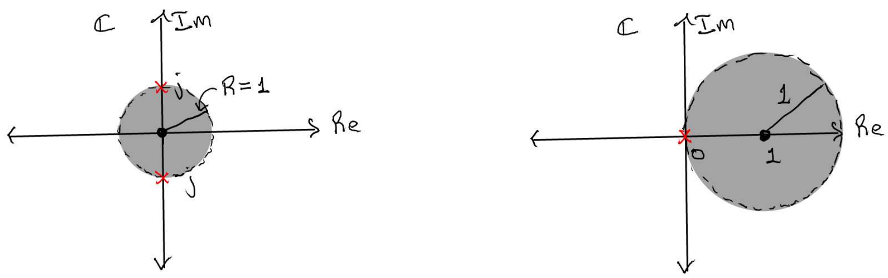
To determine the radius of convergence of this series, we may note that the series for $1/(1-w)$ is valid only in the region where $|w| < 1$. Since we set $w = -z^2$, we see that the Taylor series for $1/(z^1+1)$ is only valid for $|-z^2| < 1$, or equivalently $|z| < 1$, so the radius of convergence is equal to 1.

Alternatively, we may note that the mapping $f(z) = 1/(z^2+1)$ is analytic everywhere except at the singularities $z = \pm j$. As the distance from $z = 0$ to the nearest singularity is 1, this is an alternate way to see that the radius of convergence for the Taylor series for $1/(z^2+1)$ centred at $z_0 = 0$ has radius of convergence equal to 1.(See Figure 12.4a.)

(iii) Similar to the previous example, we may use the expansion $1/(1-w)$ about $w = 0$, but were we need to express $f(z)$ as a function of $(z-1)$ so that we may expand the Taylor series about $z_0 = 1$. (Note that $1/z$ is analytic at $z = 1$ so it will have a valid Taylor series there.) If we set $w = -(z-1)$, we get

$$
\begin{aligned}
\frac{1}{z} = \frac{1}{1-(-1)(z-1)} = \frac{1}{1-w} &= 1 + w + w^2 + w^3 + \cdots \\
&= 1 + (-1)(z-1) + (-1)^2(z-1)^2 + (-1)^3(z-1)^3 + \cdots \\
&= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n (z-1)^n
\end{aligned}
$$

which has radius of convergence equal to 1. (See Figure 12.4b.)



(a) Disk of convergence for $f(z) = 1/(z^2+1)$ about $z = 0$. The nearest singularities are at $z = \pm j$.

(b) Disk of convergence for $f(z) = 1/z$ about $z = 1$. The nearest singularity is at $z = 0$.

Figure 12.4: Radius of convergence for the examples in Example 12.5.

12.1.3 Laurent series

Consider the mapping defined by

$$f(z) = \frac{e^z}{z^3}.$$

Although $1/z^3$ is *not* analytic at $z = 0$, the exponential part e^z *is* analytic at $z = 0$ so it is not too unreasonable to consider expanding out the Taylor series of e^z and multiplying it term by term by $1/z^3$ to obtain the expression

$$
\begin{aligned}
f(z) = \frac{e^z}{z^3} &= \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) \\
&= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!} + \frac{z}{4!} + \cdots
\end{aligned}
\tag{12.6}
$$

Strictly speaking, the infinite sum in (12.6) is *not* a Taylor series of f about $z_0 = 0$, as it contains negative powers of z in its first few terms. Moreover, f cannot even *have* a valid Taylor series about $z_0 = 0$ since the function is not even defined there. Nonetheless, the infinite sum in (12.6) still converges for all $z \neq 0$, and it is useful to consider sums of this form. Before continuing, there are a few useful definitions we should introduce.

Definition 12.6. The *annulus* centred at a point z_0 with inner radius R_0 and outer radius R_1 is the set

$$D = \{z \in \mathbb{C} \,\big|\, R_0 < |z - z_0| < R_1\}.$$

Intuitively, it is the set of all points z whose distance from z_0 is *between* R_0 and R_1.

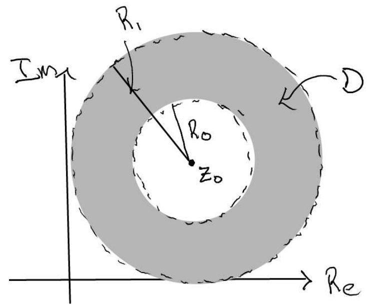


Figure 12.5: A depiction of an annulus D centred at z_0 with inner radius R_0 and outer radius R_1.

In general, sums of the form in (12.6) will not be valid on *disks* (as Taylor series' are), but will instead be valid in regions that are *annulus*-shaped. Infinite series of the type in (12.6) that have negative powers of $(z - z_0)$ are called *Laurent series*.

Definition 12.7. A *Laurent series* about a point z_0 is a doubly infinite series of the form

$$\begin{aligned}\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n &= \sum_{n=1}^{\infty} c_{-n} \frac{1}{(z - z_0)^n} + \sum_{n=0}^{\infty} c_n (z - z_0)^n \\ &= \cdots + c_{-2} \frac{1}{(z - z_0)^2} + c_{-1} \frac{1}{z - z_0} + c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \cdots .\end{aligned}$$

A Laurent series is distinctively different from a Taylor series, since a Taylor series does not allow for negative powers of $z - z_0$. (However, if a Laurent series does not contain any negative powers, then it is simply a Taylor series.) Neither of the "infinite" sums (the part of with the negative powers or the part with the positive powers) needs to actually be infinite. For example, the series in (12.6) is a Laurent series that only has three nonzero terms with negative powers.

Similar to Taylor series, Laurent series are also useful representations of functions but this time about points that are possibly singularities of f rather than only allowed to be centered at points where f is analytic. Laurent's Theorem below shows us the usefulness of Laurent series and indicates how to find the coefficients.

Theorem 12.8 (Laurent's Theorem)**.** *Let f be a mapping that is analytic everywhere inside an annulus D centred at a point z_0 with inner radius R_0 and outer radius R_1. Then f admits a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

that is valid for all $z \in D$ with coefficients given uniquely by the formula

$$c_n = \frac{1}{2\pi j} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{12.7}$$

where Γ is any positively oriented closed simple contour in D that goes around z_0.

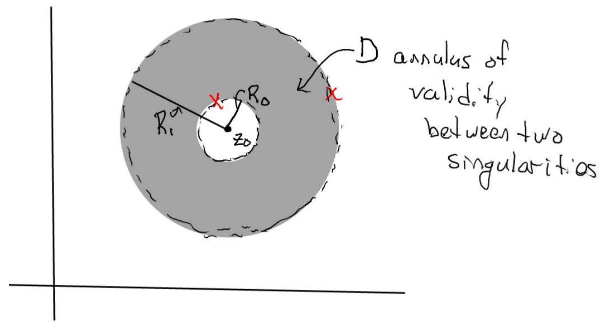


Figure 12.6: An annulus of validity for one of the Laurent series expansions of a mapping f centred at z_0.

The formula in (12.7) is in general not very useful for determining the Laurent series expansion for a given function f. For example, the Laurent series for the mapping $f(z) = e^z/z^3$ is simply given by the formula in (12.6), which we found by multiplying each term of the Taylor series for e^z by $1/z^3$. Practically speaking, we are usually able to avoid using (12.7) in developing Laurent series, as illustrated in the following examples.

Example 12.9. Find the Laurent series expansion for the mapping $f(z) = e^{1/z}$ about $z = 0$.

Solution. Similar to our approach to Taylor series, we can use the known Taylor series expansion for e^w and make the substitution $w = 1/z$ to find

$$\begin{aligned} f(z) = e^{1/z} = e^w &= 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \\ &= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots . \end{aligned} \tag{12.8}$$

In this case, the Laurent series has infinitely many terms in its expansion with negative powers. Since $1/z$ is analytic everywhere except at $z = 0$ and the Taylor series expansion for e^w is valid everywhere, the Laurent series in (12.8) is valid for all $z \neq 0$. In this case, the "annulus of validity" of this Laurent series is the region where $0 < |z|\infty$ (i.e., the annulus with inner radius 0 and outer radius ∞).

Example 12.10. Obtain all possible Laurent series expansions of

$$f(z) = \frac{1}{z(z-1)}$$

about $z_0 = 0$.

Solution. There are two points where f is singular: at $z = 0$ and $z = 1$. Hence two possible annuli around $z_0 = 0$ where f is analytic are the regions (see Figure 12.7)

(i) $\{z \,|\, 0 < |z| < 1\}$ and

(ii) $\{z \,|\, 1 < |z| < \infty\}$.

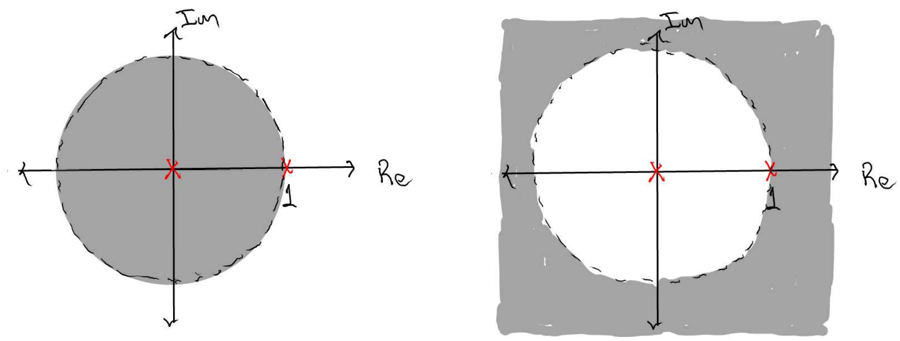


Figure 12.7: The two different 'annuli' centred at $z = 0$ where the mapping $f(z) = 1/z(z-1)$ has a valid Laurent series expansion.

We'll consider each region separately.

(i) In the region where $0|z|1$, we have the Taylor series expansion

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots . \tag{12.9}$$

Multiplying this term-by-term with $-1/z$, we obtain

$$\begin{aligned} f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} &= -\frac{1}{z}\left(1 + z + z^2 + z^3 + \cdots\right) \\ &= -\frac{1}{z} - 1 - \frac{1}{z} - \frac{1}{z^2} - \cdots . \end{aligned} \tag{12.10}$$

Since $1/z$ is analytic whenever $0 < |z|$ and the expansion in (12.9) is valid only when $|z| < 1$, we see that the Laurent series for $f(z)$ in (12.10) is valid in the annulus $\{z \,|\, 0 < |z| < 1\}$.

(ii) To determine the Laurent series expansion that is valid in the region where $1|z|\infty$, it will be useful to express $f(z)$ as a function of $1/z$,

$$\begin{aligned} f(z) = \frac{1}{z(z-1)} = \frac{1}{z}\frac{1}{z(1-\frac{1}{z})} &= \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} \\ &= \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots , \end{aligned} \tag{12.11}$$

where we make use of the known Laurent expansion

$$\frac{1}{1-\frac{1}{z}} = \frac{1}{1-w} = 1 + w + w^2 + w^3 + \cdots = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

(where we set $w = 1/z$) that is valid for $|w| < 1$, or equivalently $|z| > 1$. Hence the Laurent series expansion in (12.11) is valid in the region where $1 < |z| < \infty$.

12.2 Residues and singularities

12.2.1 Classification of singularities

Recall that if a mapping f is not analytic (or not even defined) at a point z_0 then it is said to be singular there. It will be useful to classify the different types of singularities that a mapping can have. We first introduce the idea of a *punctured disk* (which is really just an annulus with inner radius 0).

Definition 12.11. The *punctured disk* centred at a point z_0 with radius R is the set

$$D = \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < R\}.$$

It is simply the disk of radius R centred at z_0 with the point z_0 removed.

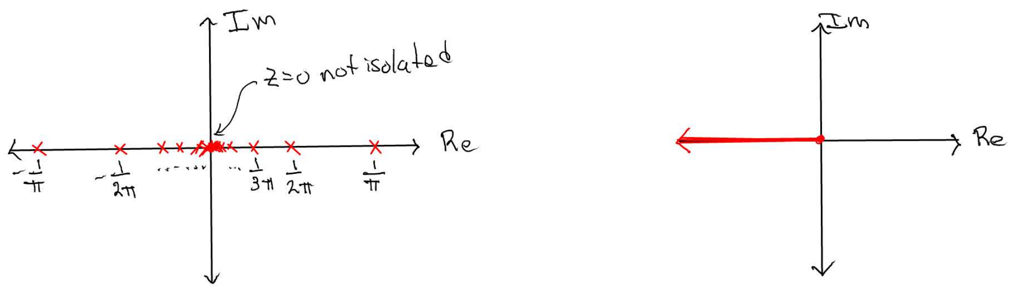
Let f be a mapping and suppose that z_0 is a singular point of f. We say that z_0 is an *isolated* singularity of f if there is some positive radius $R > 0$ such that f is analytic everywhere inside the pictured disk of radius R centred at z_0. Otherwise the singularity is said to be *non-isolated*. For example, the singularity $z_0 = 0$ of the mapping $f(z) = 1/z$ is isolated, as the mapping is analytic everywhere except at $z = 0$.

Example 12.12. We'll next consider a few examples of *non*-isolated singular points.

(i) Consider the function defined by

$$f(z) = \frac{1}{\sin \frac{1}{z}}$$

which is defined and differentiable everywhere except at $z = 0$ and wherever $\sin(1/z) = 0$. That is, it has singularities whenever $1/z = k\pi$ for some integer $k \in \mathbb{Z}$. Hence f has singularities at the points $z = 1/k\pi$ for each nonzero integer $k \in \mathbb{Z}$ because $\sin(1/z) = 0$ at those points. Each of the points $z = 1/k\pi$ is an isolated singularity of f, but the point $z = 0$ is *not* because every punctured disk $\{z \,|\, 0 < |z| < R\}$ contains at least one other singular point (in fact, an infinite number of them) no matter how small we choose R to be.(See Figure 12.8a.)



(a) The singular points of $f(z) = 1/\sin(1/z)$. The point $z = 0$ is a non-isolated singular point. Each of the other singularities at $z = 1/k\pi$ for nonzero integers k is isolated.

(b) The singular points of $f(z) = \operatorname{Log} z$. Each point along the negative real axis $x \in (-\infty, 0]$ is a non-isolated singularity of Log. The function Log is analytic everywhere else.

Figure 12.8: The non-isolated singularities of the mappings in Example 12.12.

(ii) The function $f(z) = \operatorname{Log} z$ is defined everywhere except $z = 0$, so $z = 0$ is a singular point of Log. However, Log is not continuous (and thus not differentiable) along the negative real axis, so each point $z = x$ with $x \in \mathbb{R}$ and $x < 0$ is also a singular point of Log. None of these singularities are isolated as they are all connected. (See Figure 12.8b.)

For the remainder of the course, we will only consider *isolated* singularities. By Laurent's Theorem, if a mapping f has an isolated singularity at a point z_0, there is necessarily a punctured disk $\{z \,|\, 0 < |z| < R\}$, with some positive radius R, on which f admits a valid Laurent series representation

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \\ &= \cdots + c_{-2}\frac{1}{(z-z_0)^2} + c_{-1}\frac{1}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \end{aligned} \tag{12.12}$$

for all z satisfying $0 < |z| < R$. If the expansion in (12.12) terminates on the left so that it is actually of the form

$$f(z) = c_{-m}\frac{1}{(z-z_0)^m}\cdots + c_{-2}\frac{1}{(z-z_0)^2} + c_{-1}\frac{1}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \tag{12.13}$$

for some integer m, then we say that z_0 is a *pole of order* m. (A 1^{st} order pole is also called a *simple pole.*) Otherwise, if the expansion does not terminate on the left (i.e., there are infinitely many negative powers of $(z-z_0)$ that appear in the Laurent series (12.12)), then z_0 is said to be an *essential singularity*.

Example 12.13. Consider the following examples.

(i) Note that $f(z) = e^z/z^3$ has a third order pole at $z = 0$ since its Laurent series expansion about $z = 0$ is

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!} + \frac{1}{4!}z + \cdots .$$

Since the largest integer m such that the z^{-m} term in the Laurent series has a nonzero coefficient is $m = 3$, the singularity of f at $z = 0$ is a pole of order 3.

(ii) Meanwhile, $f(z) = e^{1/z}$ has an essential singularity at $z = 0$ since its Laurent series expansion

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$$

has infinitely many nonzero terms with negative exponents.

(iii) Consider now the mapping $f(z) = 1/z(z-1)$, which has two Laurent series expansions (see Example 12.10):

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots \qquad \text{valid in the region where } 0 < |z| < 1 \tag{12.14}$$

$$\text{and} \qquad f(z) = \cdots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} \qquad \text{valid in the region where } 1 < |z| < \infty. \tag{12.15}$$

While it may appear from the Laurent series expansion in (12.15) that $z_0 = 0$ is an essential singularity, we must consider that the definition of the order of the pole depends only on the Laurent series that is valid in a punctured disk around $z_0 = 0$. The proper Laurent series to consider when determining the order of the pole is therefore the one in (12.14), which would indicate that $z_0 = 0$ is a first-order pole of f (i.e., a simple pole).

Suppose a mapping f has an isolated singularity at z_0. Normally, if z_0 is a singularity of a mapping f, the limit

$$\lim_{z\to z_0} f(z)$$

does not exist. For example, if z_0 is a pole with order $m \geq 1$ or an essential singularity, then the limit of $f(z)$ as $z \to z_0$ blows up, as

$$\lim_{z\to z_0} \frac{1}{(z-z_0)^m} \quad \text{does not exist}$$

for any integer $m \geq 1$. On the other hand, if the Laurent series expansion of f in a punctured disk around z_0 has *no* negative powers (i.e., the point z_0 is a zero$^{\text{th}}$ order pole) such that the Laurent series is of the form

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots,$$

then the Laurent series is actually just a regular Taylor series. As f is not necessarily defined at $z = z_0$ we cannot simply plug $z = z_0$ into the above expression. We may however take the limit of both sides to find

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left(c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots\right) = c_0.$$

That is, the limit of $f(z)$ as $z \to z_0$ exists and is equal to the zero$^{\text{th}}$ coefficient of the Laurent series. In this case, we say that the singularity is *removable*, since we might as well "plug the hold" with the limiting value.

Example 12.14. Consider the mapping defined by

$$f(z) = \frac{\sin z}{z}$$

for all $z \neq 0$. This is analytic everywhere it is defined, but we cannot blindly plug $z = 0$ into the formula since we cannot divide by zero. However, examining the Laurent series expansion of f at $z_0 = 0$,

$$\begin{aligned} f(z) = \frac{1}{z}\sin z &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots, \end{aligned}$$

has no negative powers of z, and thus

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right) = 1.$$

Thus $\sin z/z$ has a remove able singularity at $z = 0$. If we define the sinc mapping as

$$\text{sinc}(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

for all $z \in \mathbb{C}$, then sinc is not only continuous everywhere, it is also analytic everywhere (in particular it is analytic at $z = 0$). That is, if we "plug the hole" where $\sin z/z$ is not defined with its limiting value, we obtain a mapping that is analytic everywhere. (That is, we have "removed" the singularity.) This explains the terminology for this type of singularity.

12.2.2 Residues

We are now ready to introduce the useful concept of a *residue* (see also Figure 12.9)

Definition 12.15. Let f be analytic in some pun cured disk D centred at a point z_0. The *residue* of f at the point z_0 is defined as

$$\text{Res}(f, z_0) = \frac{1}{2\pi j} \oint_\Gamma f(z)\, dz$$

where Γ is any positively oriented simple closed contour in D going around z_0.

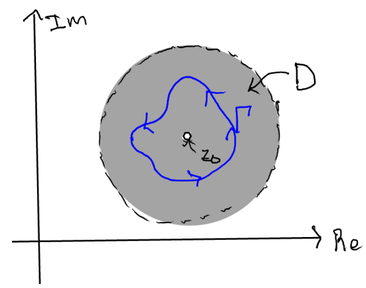


Figure 12.9: If f is analytic in some punctured disk D around a point z_0, then the integral around any positively oriented closed simple contour Γ in D that goes around z_0 will yield the same value. The unique value of this integral is proportional to the *residue* of f at z_0.

Suppose now that f is analytic in some region $D \subseteq \mathbb{C}$ except at possible some finite number of isolated singularities, and let Γ be a positively oriented simple closed contour in D (that does not cross any singularities of f). What is the value of

$$\oint_\Gamma f(z)\,dz?$$

We know from last week's notes that the value will only depend on the singularities of f that are located inside of the contour. By the principle of path deformation, we may deform the path Γ into a bunch of smaller contours $\Gamma_1, \ldots, \Gamma_N$ that go around each of the singularities $z_1, \ldots, z_N$ individually (see Figure 12.10).

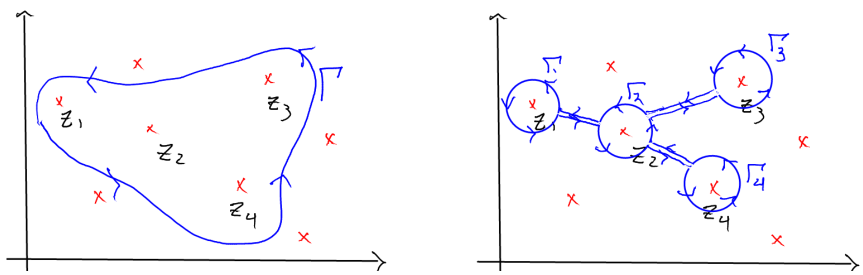


Figure 12.10: The integral of f along a simple closed contour Γ is equal to the sum of the integrals of f around each of the contours $\Gamma_1, \ldots, \Gamma_N$ that each go around each of the singularities enclosed within Γ separately.

The integral over the parts of the contour connecting each of the contours $\Gamma_1, \ldots, \Gamma_N$ cancel out (since they are integrated over twice, once forward and once backwards). Thus, we may compute the desired integral as

$$\oint_\Gamma f(z)\,dz = \sum_{k=1}^{N} \oint_{\Gamma_k} f(z)\,dz,$$

and it remains only to determine the values of the integrals along the contours $\Gamma_1, \ldots, \Gamma_N$. Since each of the singularities $z_1, \ldots, z_N$ is isolated, there is some punctured disk around each one where f is analytic, and we therefore have that

$$\oint_{\Gamma_k} f(z)\,dz = 2\pi j\,\mathrm{Res}(f, z_k)$$

for each $k \in \{1, \ldots, N\}$. Putting all of this together gives us the following important result in complex analysis.

Theorem 12.16 (Cauchy's residue theorem)**.** *Let f be analytic everywhere in a region simply connected region D except at finitely many isolated singularities in D. If Γ is any positively oriented simple closed contour in D that does not cross any of those singularities, then*

$$\oint_{\Gamma} f(z)\,dz = 2\pi j \sum_k \operatorname{Res}(f, z_k)$$

where the sum is taken over all singularities z_k that are located inside the region enclosed by the contour Γ.

The name "residue" comes from the following idea. A closed contour integral only depends on the values of the residues at each of the singular points inside that contour. We can completely ignore any of the points inside Γ that are analytic. Only at the singular points does the function leave something behind" that is picked up by the contour integral. Hence, the function leaves a "residue" at each singular point but doesn't leave anything behind at the analytic points inside Γ.

What is the point of introducing this new term? Well, if we have clever ways of computing the residues of functions at singular points, then we can make use of Cauchy's residue theorem to greatly simplify the computation of any contour integral!

12.2.3 Computing residues

If a mapping f is analytic at a point z_0, then there is some disk around z_0 where f is differentiable. The value of any closed contour integral in that disk that goes around z_0 will be zero, by the Cauchy-Goursat theorem. We arrive at the first important rule of residues:

$$\boxed{\text{if } f \text{ is analytic at } z_0\text{, then } \operatorname{Res}(f, z_0) = 0}$$

Residues get a bit more interesting at singularities. If a mapping f is of the form

$$f(z) = \frac{g(z)}{(z - z_0)^{n+1}} \tag{12.16}$$

for some integer $n \geq 0$ and some other mapping g that is analytic at z_0, then we may use the Cauchy Integral Formula to compute the residue:

$$\operatorname{Res}(f, z_0) = \operatorname{Res}\left(\frac{g(z)}{(z - z_0)^{n+1}},\, z_0\right) = \frac{1}{2\pi j}\oint_{\Gamma} \frac{g(z)}{(z - z_0)^{n+1}}\,dz = \frac{1}{n!} g^{(n)}(z_0)$$

where Γ is any positively oriented simple closed contour going around z_0 such that f is analytic everywhere inside Γ except possibly at z_0.

What if f is not of the nice form in (12.16)? In this case we can not just apply Cauchy's Integral Formula, but we can use our knowledge of Laurent series to help us out. If z_0 is an isolated singularity of f, then it has a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n$$

that is valid in some punctured disk around z_0. Taking the integral around any positively oriented simple closed contour Γ in that punctured disk around z_0 gives us

$$\begin{aligned}
\operatorname{Res}(f, z_0) &= \frac{1}{2\pi j}\oint_\Gamma f(z)\,dz \\
&= \frac{1}{2\pi j}\oint_\Gamma \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n\,dz \\
&= \sum_{n=-\infty}^{+\infty} \frac{c_n}{2\pi j}\oint_\Gamma (z-z_0)^n\,dz \\
&= \sum_{n=-\infty}^{-2}\left(\frac{c_n}{2\pi j}\underbrace{\oint_\Gamma (z-z_0)^n\,dz}_{=0}\right) + \frac{c_{-1}}{2\pi j}\underbrace{\oint_\Gamma \frac{1}{z-z_0}\,dz}_{=2\pi j} + \sum_{n=0}^{+\infty}\left(\frac{c_n}{2\pi j}\underbrace{\oint_\Gamma (z-z_0)^n\,dz}_{=0}\right) \\
&= 0 + c_{-1} + 0 = c_{-1},
\end{aligned}$$

where we integrate separately over each term of the infinite Laurent series and make use of our well-known integral

$$\oint_\Gamma (z-z_0)^n\,dz = \begin{cases} 2\pi j & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

for all integers $n \in \mathbb{Z}$ and all positively oriented simple closed contours going around z_0. In particular, we see that integrating $f(z)$ along some contour going around z_0 picks out exactly one coefficient of the infinite series: the coefficient of the $(z-z_0)^{-1}$ term. This has actually quite remarkable implications; if we can determine the value of the coefficient of the (-1)-term of the Laurent series expansion of f then we have found the value of the residue! Since this is such an important result, we'll put it inside of its own box:

> If z_0 is a singular point of f and f has a Laurent series expansion
>
> $$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n$$
>
> that is valid in some punctured disk around z_0, then
>
> $$\operatorname{Res}(f, z_0) = c_{-1}. \tag{12.17}$$

Example 12.17. Consider the mapping defined by

$$f(z) = e^{1/z} + \frac{e^{2z}}{z^3}.$$

Compute the integral

$$\oint_\Gamma f(z)\,dz$$

where Γ is the unit circle centered at the origin oriented counterclockwise.

Solution. Note that f has only one singularity at $z = 0$. It will first be helpful to notice that

$$\operatorname{Res}(g(z) + h(z), z_0) = \operatorname{Res}(g(z), z_0) + \operatorname{Res}(h(z), z)$$

if g and h are functions that are both analytic in some punctured disk around z_0, so we may find the residues of each of the terms of f separately. We have the Laurent series expansions

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots && \text{with } c_{-1} = 1 \\ \text{and} \quad \frac{e^{2z}}{z^3} &= \frac{1}{z^3}\left(1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \frac{2^4 z^4}{4!} + \cdots\right) \\ &= \frac{1}{z^3} + \frac{2}{z^2} + 2\frac{1}{z} + \frac{4}{3} + \frac{2}{3}z + \cdots && \text{with } c_{-1} = 2, \end{aligned}$$

and thus

$$\operatorname{Res}\left(e^{1/z}, 0\right) = 1 \qquad \text{and} \qquad \operatorname{Res}\left(\frac{e^{2z}}{z^3}, 0\right) = 2.$$

The desired integral is therefore

$$\begin{aligned} \oint_\Gamma f(z)\,dz = 2\pi j \operatorname{Res}(f, 0) &= 2\pi j \left(\operatorname{Res}\left(e^{1/z}, 0\right) + \operatorname{Res}\left(\frac{e^{2z}}{z^3}, 0\right)\right) \\ &= 2\pi j(1 + 2) = 6\pi j. \end{aligned}$$

12.2.3.1 Residues of simple poles

If a mapping f has a simple pole at a point z_0, then it possesses a Laurent series of the form

$$f(z) = c_{-1}\frac{1}{z - z_0} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots \tag{12.18}$$

that is valid in some punctured disk around z_0 for some nonzero value of the coefficient c_{-1}. The residue of f at z_0 will be $\operatorname{Res}(f, z_0) = c_{-1}$, but if we don't know the value of this coefficient, how might we find it? If we multiply the entire expression in (12.18) by $(z - z_0)$, we obtain

$$(z - z_0)f(z) = c_{-1} + c_0(z - z_0) + c_1(z - z_0)^2 + c_2(z - z_0)^3 + \cdots \tag{12.19}$$

which now has *no* negative powers of $(z - z_0)$ appearing. That is, the expression $(z - z_0)f(z)$ has a remove able singularity at the point z_0 with limiting value

$$\begin{aligned} \lim_{z\to z_0}\big((z - z_0)f(z)\big) &= \lim_{z\to z_0}\big(c_{-1} + c_0(z - z_0) + c_1(z - z_0)^2 + c_2(z - z_0)^3 + \cdots\big) \\ &= c_{-1} \\ &= \operatorname{Res}(f, z_0). \end{aligned}$$

This gives us a useful formula for finding the residues of simple poles:

> If a mapping f has a simple pole at z_0, then
>
> $$\operatorname{Res}(f, z_0) = \lim_{z\to z_0}\big((z - z_0)f(z)\big) \tag{12.20}$$

How might we determine when a singularity is a simple pole? Well, if z_0 is a pole with order $m \geq 1$, then the limit

$$\lim_{z\to z_0} f(z)$$

does not exist (otherwise the singularity would be remove able). On the other hand, if the order of the pole is $m \geq 2$, then the limit

$$\lim_{z\to z_0}\big((z-z_0)f(z)\big) = \lim_{z\to z_0}\big(c_{-m}\frac{1}{(z-z_0)^{m-1}} + \cdots + c_{-1} + c_0(z-z_0) + c_1(z-z_0)^2 + \cdots\big)$$

would not exist, since the part of the expression $1/(z-z_0)^{m-1}$ would blow up as $z \to z_0$. Thus, if

$$\lim_{z\to z_0} f(z) \text{ does not exist} \qquad \text{but} \qquad \lim_{z\to z_0}\big((z-z_0)f(z)\big) \text{ does exist,}$$

then z_0 is a simple pole (i.e., pole of order 1) of f and its residue can be computed using (12.20).

Example 12.18. Show that the following mappings have simple poles at $z = 0$ and find the residues there.

(i) $f(z) = \dfrac{z}{1-\cos z}$

Solution. We may use l'Hopital's rule to find that the limit

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{z}{1-\cos z} = \lim_{z\to 0}\frac{1}{\sin z}$$

does not exist. However, by repeated use of l'Hopital's rule we find that

$$\lim_{z\to 0}\big((z)f(z)\big) = \lim_{z\to 0}\frac{z^2}{1-\cos z} = \lim_{z\to 0}\frac{2z}{\sin z} = \lim_{z\to 0}\frac{2}{\cos z} = 2,$$

and thus 0 is a simple pole of $f(z) = z/(1-\cos z)$ with $\operatorname{Res}(f,0) = 2$.

(i) $f(z) = \dfrac{1}{z(1-z)}$

Solution. We can clearly see that 0 is a simple pole of f, and thus

$$\operatorname{Res}(f,0) = \lim_{z\to 0}\big(zf(z)\big) = \lim_{z\to 0}\left(\frac{1}{1-z}\right) = 1.$$

Remark 12.19. Suppose that g and h are mappings that are both analytic at z_0 such that $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$, and consider the mapping defined by

$$f(z) = \frac{g(z)}{h(z)}.$$

We see that z_0 is an isolated singularity of f, but it is also a simple pole of f with residue

$$\begin{aligned}\lim_{z\to z_0}\left((z-z_0)\frac{g(z)}{h(z)}\right) &= g(z_0)\lim_{z\to z_0}\frac{(z-z_0)}{h(z)}\\ &= g(z_0)\frac{1}{\lim_{z\to z_0}\left(\frac{h(z)-h(z_0)}{z-z_0}\right)} = \frac{g(z_0)}{h'(z_0)},\end{aligned}$$

where we make use of the assumption that $h(z_0) = 0$ and use the definition of the derivative. We therefore see that z_0 is a simple pole of f with residue $\operatorname{Res}(f,z_0) = \frac{g(z_0)}{h'(z_0)}$. This fact is also useful to box:

$$\boxed{\operatorname{Res}(f,z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{where} \quad f(z) = \frac{g(z)}{h(z)} \quad \text{if } g(z_0), h'(z_0) \neq 0 \text{ and } h(z_0) = 0.} \tag{12.21}$$

Example 12.20. Show that $f(z) = 1/(z^3 - 1)$ has a simple pole at $z = 1$ and compute the residue there.

Solution. Here we note that f is of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z) = 1$ and $h(z) = z^3 - 1$. Since the limit if $f(z)$ as $z \to 0$ is not defined but

$$\frac{g(1)}{h'(1)} = \frac{1}{3}$$

as $h'(z) = 3z$, we see that 1 is indeed a simple pole of f and that $\operatorname{Res}(f, 1) = 1/3$.

12.2.3.2 Higher order poles

Suppose now that a mapping f has a pole of order m at a point z_0 such that f has a Laurent series of the form

$$f(z) = c_{-m}\frac{1}{(z-z_0)^m} + \cdots + c_{-2}\frac{1}{(z-z_0)^2} + c_{-1}\frac{1}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots.$$

If we do not know the coefficients, how might we get the c_{-1} coefficient to pop out? Multiplying the expression this time by $(z-z_0)^m$ gives us

$$(z-z_0)^m f(z) = c_{-m} + \cdots + c_{-2}(z-z_0)^{m-2} + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + c_1(z-z_0)^{m+1} + \cdots.$$

If we take the $(m-1)^{\text{th}}$ derivative of this expression and take the limit as $z \to z_0$, we get

$$\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big)\right] = \cdots = (m-1)! c_{-1}$$

(where we have omitted some of the details, but it is easy to check). This gives us another useful rule that we will box:

> If a mapping f has a pole of order m at z_0, then
>
> $$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!}\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big)\right]. \tag{12.22}$$

Example 12.21. Compute the reside of

$$f(z) = \frac{z^3 + 2z}{(z-j)^3}$$

at $z = j$.

Solution. This mapping clearly has a third-order pole at $z = j$, so we may use the formula in (12.22) to find

$$\begin{aligned}
\operatorname{Res}(f, j) &= \frac{1}{2!}\lim_{z \to j}\left[\frac{d^2}{dz^2}\big((z-j)^2 f(z)\big)\right] \\
&= \frac{1}{2}\lim_{z \to j}\Bigg[\underbrace{\frac{d^2}{dz^2}(z^3 + 2z)}_{6z}\Bigg] = \frac{1}{2}\lim_{z \to j}(6z) \\
&= \frac{6j}{2} = 3j.
\end{aligned}$$

12.3 Applications of residue theory

12.3.1 Integrals of functions of sines and cosines

(We didn't have time to cover this part!)

12.3.2 Real integrals from $-\infty$ to ∞

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a *real*-valued function on the real line. We can use complex analysis and residue theory to help us compute certain types of integrals that can't normally be computed so easily. To do so, we first introduce an important tool. It is often useful to "bound" the value of a complex integral by obtaining an upper limit to the modulus of its value.

Let f be a mapping and let Γ be a contour in a region where f is defined. If L is the length of Γ and M is a number such that $|f(z)| \leq M$ holds for all points z on the contour Γ, then

$$\boxed{\left| \int_\Gamma f(z)\, dz \right| \leq ML.} \tag{12.23}$$

This result is called the *ML-bound*.

Example 12.22. Use the ML-bound to show that

$$\left| \oint_\Gamma \frac{\sin z}{z(z^2+9)}\, dz \right| \leq \frac{\pi\sqrt{\cosh 10}}{2},$$

where Γ is the circular contour of radius 5 centered at the origin (see the figure below).

Solution. First note that for $z = x + jy$ we can split up $\sin z$ into its real and imaginary parts as

$$\sin z = \sin(x + jy) = \sin x \cosh y + j \cos x \sinh y.$$

Note that $|x| \leq 5$ and $|y| \leq 5$ for points $z = x + jy$ on the circle Γ of radius 5. Thus

$$\begin{aligned} |\sin(x+jy)| &= \sqrt{\sin^2 x \cosh 2y + \cos^2 x \sinh^2 y} \\ &\leq \sqrt{\cosh 2y + \sinh^2 y} && (\text{since } \sin^2 x \leq 1 \text{ and } \cos^2 x \leq 1) \\ &\leq \sqrt{\cosh 25 + \sinh^2 5} = \sqrt{\cosh 10}, && (12.24) \end{aligned}$$

where the last inequality in (12.24) follows from the identity $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$. Meanwhile, it holds that $|z| = 5$ and

$$|9 + z^2| \geq 9 - |z|^2 = 9 - 5 = 4.$$

for all z on Γ, where the inequality follows from the fact that $|\alpha| + |\beta| \geq |\alpha - \beta|$ for all $\alpha, \beta \in \mathbb{C}$. It follows that

$$\left| \frac{\sin z}{z(9+z^2)} \right| \leq \frac{\sqrt{\cosh 10}}{(5)(4)} = \frac{\sqrt{\cosh 10}}{20}$$

for all z on Γ, so we may choose $M = \sqrt{\cosh 10}/20$. As Γ is a circle of radius 5, it has length $L = 10\pi$, so we may use the ML-estimation technique to bound

$$\left| \oint_\Gamma \frac{\sin z}{z(z^2+9)}\, dz \right| \leq ML = \frac{\sqrt{\cosh 10}}{20} 10\pi = \frac{\pi\sqrt{\cosh 10}}{2}.$$

A more useful application of the ML-bound is to use it to show that the value of some integral goes to zero in some limit. In particular, if Γ_R is some contour that depends on some parameter R such that, for some complex mapping f, the modulus of the integral of f tends to zero as R tends to infinity, then it must be the case that the integral itself goes to zero (and not just the modulus). That is,

$$\text{if} \quad \lim_{R\to\infty} \left| \int_{\Gamma_R} f(z)\,dz \right| = 0 \qquad \text{then} \quad \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0.$$

This is incredibly useful for computing certain types of *real* integrals with infinite limits of integration, as the following few examples show.

Example 12.23. Compute the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx \tag{12.25}$$

using residue theory.

Solution. First note that the infinite limits of integration should actually be taken as a limit,

$$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \lim_{R\to\infty} \int_{-R}^{+R} \frac{1}{x^6+1}\,dx.$$

Next, note that we can extend the function $f(x) = 1/(x^6+1)$ to a mappings of complex numbers that takes complex values as inputs. If we let Γ_R be the contour in the complex plane along the real axis from $-R$ to R, we see that the original real integral can be viewed as a complex contour integral

$$\int_{-R}^{+R} \frac{1}{x^6+1}\,d = \int_{\Gamma_R} \frac{1}{z^6+1}\,dz.$$

To make use of residue theory, we can add an additional contour C_R to be the semicircular contour of radius R in the upper half-plane that connects $+R$ to $-R$ (see Figure 12.11). We can now compute the integral around the resulting closed contour

$$\oint_{\Gamma_R \cup C_r} \frac{1}{z^6+1}\,dz \tag{12.26}$$

using reside theory!

To compute (12.26) we must first locate the singularities of the integrand and compute the residues there. The roots of the polynomial z^6+1 are all of the sixth roots of -1. Since $e^{j\pi + jk2\pi} = -1$ for all integers $k \in \mathbb{Z}$, the sixth roots of -1 are of the form

$$(-1)^{1/6} = e^{j\pi(1+2k)/6} = z_k \qquad \text{for } k = 0,1,2,3,4,5.$$

The roots $z_0, z_1, \ldots, z_5$ are equally spaced on the unit circle, with z_0, z_1, z_2 located above the real axis and z_3, z_4, z_5 located below the real axis (see Figure 12.11). More explicitly, they are

$$z_0 = e^{j\pi/6}, \quad z_1 = e^{j\pi/2}, \quad z_2 = e^{j5\pi/6}, \quad z_3 = e^{j7\pi/6}, \quad z_4 = e^{j3\pi/2}, \quad \text{and} \quad z_5 = e^{j11\pi/6}.$$

If R is big enough, the closed contour $\Gamma_R \cup C_r$ contains each of the singularities as z_0, z_1, and z_2. Note that $h'(z) = 6z^5$ for the function $h(z) = z^6+1$, so we may use the rule in (12.21) to compute the residues at the singularities as

$$\operatorname{Res}\left(\frac{1}{z^6+1}, z_k\right) = \frac{1}{6z_k^5}.$$

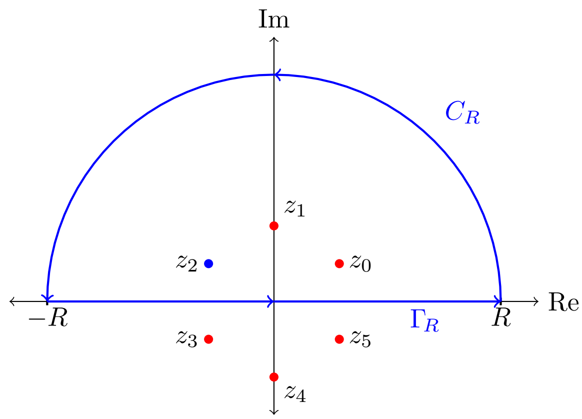


Figure 12.11: The singularities of $1/(z^6+1)$ are indicated in red. The closed contour $\Gamma_R \cup C_R$ used in Example 12.25 consisting of the segment Γ_R of the real axis from $-R$ to $+R$ and the semicircular arc C_R of radius R going counterclockwise in the upper half-plane, and it encloses the singularities at z_0, z_1, and z_2.

Hence the residues of the integrand at z_0, z_1, and z_2 are

$$z_0: \qquad \mathrm{Res}\left(\frac{1}{z^6+1},\, z = e^{j\pi/6}\right) = \frac{1}{6\left(e^{j\pi/6}\right)^5} = \frac{1}{6}e^{-j5\pi/6} = \frac{1}{6}\left(-\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)$$

$$z_1: \qquad \mathrm{Res}\left(\frac{1}{z^6+1},\, z = e^{j\pi/2}\right) = \frac{1}{6\left(e^{j\pi/2}\right)^5} = \frac{1}{6}e^{-j5\pi/2} = \frac{1}{6}e^{-j\pi/2} = -j\frac{1}{6}$$

$$z_2: \qquad \mathrm{Res}\left(\frac{1}{z^6+1},\, z = e^{j5\pi/6}\right) = \frac{1}{6\left(e^{j5\pi/6}\right)^5} = \frac{1}{6}e^{-j25\pi/6} = \frac{1}{6}e^{-j\pi/6} = \frac{1}{6}\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right).$$

We may add up the residues to compute the contour integral

$$\begin{aligned}\oint_{\Gamma_R \cup C_R} \frac{1}{z^6+1}\,dz &= 2\pi j\left(\mathrm{Res}\left(\frac{1}{z^6+1}, e^{j\pi/6}\right) + \mathrm{Res}\left(\frac{1}{z^6+1}, e^{j\pi/2}\right) + \mathrm{Res}\left(\frac{1}{z^6+1}, e^{j5\pi/6}\right)\right)\\ &= 2\pi j\frac{1}{6}\left(\left(-\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) - j + \left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\right)\\ &= \frac{2\pi}{3}.\end{aligned}$$

As long as $R > 1$, this integral does not depend on the value of R. Now, integrating along Γ_R (the segment of the real axis from $-R$ to $+R$) gives us the same value as if we had integrated along the closed loop $\Gamma_R \cup C_R$ and subtracted the result from integrating along the semicircular contour C_R. That is,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \lim_{R\to\infty}\int_{-R}^{+R} \frac{1}{x^6+1}\,dx &= \lim_{R\to+\infty}\int_{\Gamma_R} \frac{1}{z^6+1}\,dz\\ &= \lim_{R\to+\infty}\left(\int_{\Gamma_R\cup C_R} \frac{1}{z^6+1}\,dz - \int_{C_R} \frac{1}{z^6+1}\,dz\right)\\ &= \frac{2\pi}{3} - \lim_{R\to\infty}\int_{C_R} \frac{1}{z^6+1}\,dz. \qquad (12.27)\end{aligned}$$

Therefore, to compute the original integral in (12.25), it remains to show that

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z^6+1}\,dz=0.$$

To do so, we will make use of the ML-bound. For points z on the semicircular contour C_R we have $|z|=R$, and thus

$$|z^6+1|\geq |z|^6-1=R^6-1.$$

Therefore for any $R>1$ we have

$$\left|\frac{1}{z^6+1}\right|\leq\frac{1}{R^6-1}$$

for all points z on C_R. As the semicircular contour C_R has length πR, we may use the ML-bound to obtain

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{1}{z^6+1}\,dz\right|\leq\lim_{R\to\infty}\frac{\pi R}{R^6-1}=0$$

from which it follows that the integral itself (without the modulus) must also go to zero

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z^6+1}\,dz=0. \tag{12.28}$$

Putting together (12.27) and (12.28) we obtain the result that

$$\int_{-\infty}^{+\infty}\frac{1}{x^6+1}\,dx=\frac{2\pi}{6}.$$

Remark 12.24. The method employed in Example 12.23 is fairly general, and may be applied to compute many integrals of the form

$$\int_{-\infty}^{\infty}f(x)\,dx \tag{12.29}$$

provided that f has finitely many singularities in the upper half-plane and the limit

$$\lim_{R\to\infty}\left|\int_{C_R}f(z)\,dz\right|=0$$

holds for the function f on the semicircular contour C_R. This will always occur provided the value of $|f(z)|$ shrinks at least as rapidly as $1/|z|^2$ as $|z|\to\infty$. In this case, integrals of the form in (12.29) can be computed using residue theory to find that

$$\int_{-\infty}^{\infty}f(x)\,dx=2\pi j\sum_{k=1}^{N}\mathrm{Res}(f,z_k),$$

where we sum over each of the singularities $z_1,\ldots,z_N$ of f in the upper half-plane. Another example of this method in action is shown below.

Example 12.25. Evaluate the integral

$$\int_{-\infty}^{+\infty}\frac{1}{x^2+2x+2}\,dx$$

by applying the method used in Example 12.23 and explained in Remark 12.24.

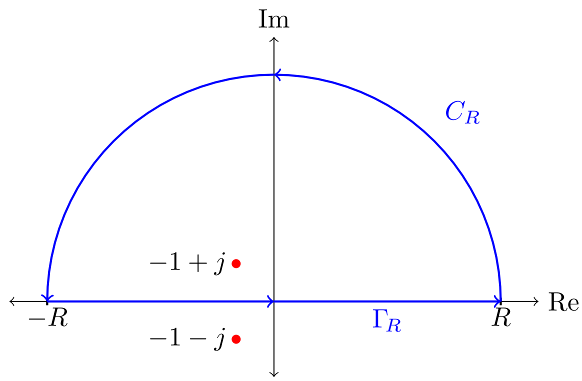


Figure 12.12: The closed contour $\Gamma_R \cup C_R$ used in Example 12.25 consisting of the segment Γ_R of the real axis from $-R$ to $+R$ and the semicircular arc C_R of radius R going counterclockwise in the upper half-plane. This closed contour encloses the singularity of $1/(z^2+2z+2)$ at $z=-1+j$.

Solution. As before, we must interpret the integral with infinite limits of integration as a limit and consider the resulting real integral as a contour integral of the mapping $f(z) = 1/(z^2+2z+1)$ along the contour on the real axis from $-R$ to R,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+2x+2}\,dx \\ &= \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{z^2+2z+2}\,dz \\ &= \lim_{R\to\infty} \left(\oint_{\Gamma_R \cup C_R} \frac{1}{z^2+2z+2}\,dz - \int_{C_R} \frac{1}{z^2+2z+2}\,dz \right).\end{aligned}$$

We may make use of residue theory to compute the contour integral along the closed loop consisting of the straight line segment Γ_R from $-R$ to $+R$ followed by the semicircular contour C_R in the upper half-plane. The integrand can be written as

$$f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z+1-j)(z+1+j)}$$

which has singularities at $-1+j$ and $-1-j$. Each of these singularities is a simple pole, but only the first of those singularities is contained in $\Gamma_R \cup C_R$. The residue of f at $z=-1+j$ is

$$\operatorname{Res}(f, -1+j) = \lim_{z\to -1+j} \big((z+1-j)f(z)\big) = \lim_{z\to -1+j} \left(\frac{1}{z+1+j}\right) = \frac{1}{j-1+1+j} = \frac{1}{2j}$$

and thus, for values of R large enough, we have

$$\oint_{\Gamma_R \cup C_R} \frac{1}{z^2+2z+2}\,dz = 2\pi j \operatorname{Res}(f, -1+j) = \pi.$$

To show that the integral along the semicircular contour C_R vanishes in the limit as $R\to\infty$, note that

$$|z^2+2z+2| \geq |z|^2 - 2|z| - 2 = R^2 - 2R - 2$$

and thus

$$\left|\frac{1}{z^2+2z+2}\right| \leq \frac{1}{R^2-2R-2}$$

for points z on C_R (i.e., $|z| = R$). As before, the length of the semicircular arc C_R is πR, and we may use the ML-bound

$$\left|\int_{C_R} \frac{1}{z^2+2z+2}\,dz\right| \leq \frac{1}{R^2-2R-2}\pi R.$$

We therefore have that

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{1}{z^2+2z+2}\,dz\right| \leq \lim_{R\to\infty}\frac{\pi R}{R^2-2R-2} = 0$$

and thus

$$\lim_{R\to\infty}\int_{C_R} \frac{1}{z^2+2z+2}\,dz = 0.$$

We may therefore conclude that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,dx = \pi.$$

12.3.3 Fourier transforms

As our final application, we will show how residue theory can be used to compute Fourier and inverse Fourier transforms. The *Fourier transform* $F = \mathcal{F}(f)$ of a function f is another function F that is defined as

$$F(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-j\omega x}\,dx$$

for all values $\omega \in \mathbb{C}$ where the integral converges. The *inverse Fourier transform* of a function F is another function $f = \mathcal{F}^{-1}(F)$ given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega x}\,d\omega$$

for all values $x \in \mathbb{R}$ where the integral converges. Using the same approach as in Section 12.3.2, we can use residue theory to compute such transforms.

Example 12.26. Determine the inverse Fourier transform of the function

$$F(\omega) = \frac{1}{\omega^2+1}.$$

Solution. Essentially, we are asked to compute the integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{j\omega x}}{\omega^2+1}\,d\omega$$

for different values of x. The integral we must compute is technically a limit

$$\int_{-\infty}^{+\infty} \frac{e^{j\omega x}}{\omega^2+1}\,d\omega = \lim_{R\to+\infty}\int_{-R}^{+R} \frac{e^{j\omega x}}{\omega^2+1}\,d\omega,$$

which we may consider as a contour integral and use residue theory to compute it. We first note that the integrand

$$\frac{e^{j\omega x}}{\omega^2+1} = \frac{e^{j\omega x}}{(\omega+j)(\omega-j)} \tag{12.30}$$

has first-order poles at $\omega = \pm j$ with residues

$$\operatorname{Res}\left(\frac{e^{j\omega x}}{(\omega+j)(\omega-j)}, \omega = j\right) = \frac{e^{j(j)x}}{((j)+j)} = \frac{e^{-x}}{2j}$$

$$\text{and}\quad \operatorname{Res}\left(\frac{e^{j\omega x}}{(\omega+j)(\omega-j)}, \omega = -j\right) = \frac{e^{j(-j)x}}{((-j)-j)} = -\frac{e^{x}}{2j}.$$

Next, note that we may expand ω in Cartesian coordinates as $\omega = u + jv$ and find that

$$e^{j\omega x} = e^{j(u+jv)x} = e^{-vx}e^{jux}.$$

In order to use the method of ML-estimates to bound the integral of (12.30), we must find an upper bound of

$$\left|\frac{e^{j\omega x}}{\omega^2+1}\right| = \left|\frac{e^{-\operatorname{Im}(\omega)x}e^{j\operatorname{Re}(\omega)x}}{\omega^2+1}\right| = \underbrace{\left|e^{j\operatorname{Re}(\omega)x}\right|}_{=1}\left|\frac{e^{-\operatorname{Im}(\omega)x}}{\omega^2+1}\right| \le \frac{e^{-\operatorname{Im}(\omega)x}}{R^2-1}$$

for points $\omega = u + vj$ on the circle $|z| = R$ (as long as $R > 1$). To do this, however, we must consider the following two cases separately: (i) the case where $x > 0$, and (ii) the case where $x < 0$.

(i) If $x > 0$, we may use the the same strategy as in the examples in Section 12.3.2, and consider the closed contour integral along the real axis from from $-R$ to $+R$ followed by the semicircular contour C_R of radius R in the upper half-plane.

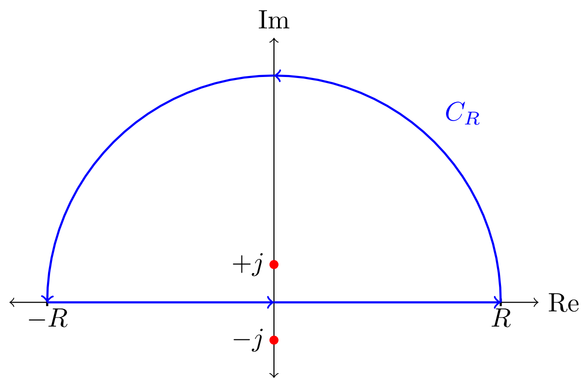


Figure 12.13: The closed contour $\Gamma_R \cup C_R$ for computing the inverse Fourier transform in case (i) (with $x > 0$) of Example 12.26, which is positively oriented and encloses the singularity at $\omega = j$.

Points $\omega = \operatorname{Re}(\omega) + j\operatorname{Im}(\omega)$ in the upper half-plane have positive imaginary part $\operatorname{Im}(\omega) \ge 0$. Since we have assume that $x > 0$ as well, note that $-\operatorname{Im}(\omega)x \le 0$ and thus $e^{-\operatorname{Im}(\omega)x} \le 1$. Hence,

$$\frac{e^{-\operatorname{Im}(\omega)x}}{R^2-1} \le \frac{1}{R^2-1}$$

and thus

$$\left|\frac{e^{j\omega x}}{\omega^2+1}\right| = \left|\frac{e^{j\operatorname{Re}(\omega)x}e^{-\operatorname{Im}(\omega)x}}{\omega^2+1}\right| \le \frac{1}{R^2-1} \tag{12.31}$$

for all ω on the semicircular contour of radius R in the upper half-plane. Therefore

$$\lim_{R\to+\infty}\left|\int_{C_R}\frac{e^{j\omega x}}{\omega^2+1}\,d\omega\right| \le \lim_{R\to+\infty}\frac{\pi R}{R^2-1} = 0 \qquad \text{and thus} \qquad \lim_{R\to+\infty}\int_{C_R}\frac{e^{j\omega x}}{\omega^2+1}\,d\omega = 0.$$

Since the closed contour that follows the part of the real line from $-R$ to $+R$ followed by the semicircular arc in the upper half-plane goes counterclockwise around the singularity at $\omega = j$, we have

$$f(x) = \frac{1}{2\pi}\lim_{R\to+\infty}\int_{-R}^{+R}\frac{e^{j\omega x}}{\omega^2+1}\,d\omega = \frac{1}{2\pi}2\pi j\underbrace{\operatorname{Res}\left(\frac{e^{j\omega x}}{(\omega+j)(\omega-j)},\omega=j\right)}_{=\frac{e^{-x}}{2j}} - \frac{1}{2\pi}\underbrace{\lim_{R\to+\infty}\int_{C_R}\frac{e^{j\omega x}}{\omega^2+1}\,d\omega}_{=0} \tag{12.32}$$

$$= j\left(\frac{e^{-x}}{2j}\right)$$

$$= \frac{e^{-x}}{2} \tag{12.33}$$

for $x > 0$.

(ii) In the case when $x < 0$, we have that $-\operatorname{Im}(\omega)x \le 0$ for points $\omega = \operatorname{Re}(\omega) + j\operatorname{Im}(\omega)$ in the upper half-plane (with $\operatorname{Im}(\omega) \ge 0$), and we can no longer use the bound in (12.31). Indeed, in the limit as $R \to \infty$, the imaginary parts of ω on the circle $|\omega| = R$ can take on larger and larger positive values, and thus $-\operatorname{Im}(\omega)x > 0$ can take on larger and larger positive values since x is negative. However, we can instead use the semicircular contour of radius R in the *lower* half-plane (see Figure 12.14) to create a different closed contour than the one used in case (i).

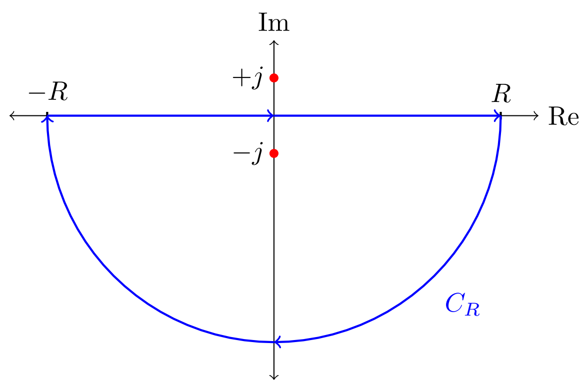


Figure 12.14: The closed contour $\Gamma_R \cup C_R$ for computing the inverse Fourier transform in case (i) (with $x > 0$) of Example 12.26, which is *negatively* oriented and encloses the singularity at $\omega = -j$.

Now our closed contour consists of the segment of the real axis from $-R$ to R follows by the semicircular arc C_R of radius R in the *lower half-plane* (and going *clockwise*). Points ω on C_R in the lower half-plane have negative imaginary component $\operatorname{Im}(\omega) \le 0$ and thus $-\operatorname{Im}(\omega)x \le 0$ (since we have assumed in this case that x is negative). Hence $e^{-\operatorname{Im}(\omega)x} \le 1$ such that

$$\left|\frac{e^{j\omega x}}{\omega^2+1}\right| = \left|\frac{e^{j\operatorname{Re}(\omega)x}e^{-\operatorname{Im}(\omega)x}}{\omega^2+1}\right| \le \frac{1}{R^2-1} \tag{12.34}$$

for all points ω on the semicircular contour C_R in the lower half-plane. As before, we now have that

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{j\omega x}}{\omega^2+1}\,d\omega = 0.$$

However, as the closed contour in Figure 12.14 now is *negatively* oriented instead of positively oriented (i.e., goes clockwise around the singularity instead of counterclockwise), the value of the closed contour integral around the singularity will be equal to the negative of $2\pi j$ times the residue. Since the singularity enclosed is now at $\omega = -j$, this is

$$\begin{aligned} f(x) = \frac{1}{2\pi} \lim_{R \to +\infty} \int_{-R}^{+R} \frac{e^{j\omega x}}{\omega^2+1}\, d\omega &= \frac{1}{2\pi}\left(\left[-2\pi j \underbrace{\operatorname{Res}\left(\frac{e^{j\omega x}}{(\omega+j)(\omega-j)}, \omega = -j\right)}_{=-\frac{e^x}{2j}}\right] - \underbrace{\lim_{R\to+\infty}\int_{C_R} \frac{e^{j\omega x}}{\omega^2+1}\, d\omega}_{=0}\right) \\ &= j\left(\frac{e^x}{2j}\right) \\ &= \frac{e^x}{2} \end{aligned} \tag{12.35}$$

for $x < 0$.

Putting together the results of (12.33) and (12.35), we find that

$$f(x) = \begin{cases} \dfrac{e^{-x}}{2}, & x > 0 \\ \dfrac{e^{x}}{2}, & x < 0, \end{cases}$$

or equivalently, $f(x) = \dfrac{e^{-|x|}}{2}$.