ECE 206 Fall 2019
Practice Problems Week 1

1. Consider the vectors $\boldsymbol{u} = (1, -2, 2)$ and $\boldsymbol{v} = (1, 1, -4)$.
   (a) Compute the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.
   (b) Compute the area of the parallelogram defined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

2. Find a parametric representation of the following curves. (Hint: use the identity $\sin^2\theta + \cos^2\theta = 1$.)
   (a) The intersection of the ellipsoid $\{(x, y, z) : x^2 + 3y^2 + 4z^2 = 1\}$ with the plane $\{(x, y, z) : y = x\}$.
   (b) The intersection of the elliptical cylinder $\{(x, y, z) : x^2 + 4y^2 = 4\}$ with the hyperboloid $\{(x, y, z) : z^2 = 2 + x + y^2\}$ in the region where $z \geq 0$.
   (c) The helix that lies on the cylinder $\{(x, y, z) : x^2 + y^2 = 4\}$ starting at $(2, 0, 0)$ and ending at $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ and going counter-clockwise around the z-axis.

3. Let $a \in \mathbb{R}$ be a constant. Give a parametric representation of the circle

$$\Gamma = \{(x, y, z) : x^2 + y^2 + z^2 = 4a^2 \text{ and } x = a\}$$

   and find its total length by integration. Check your answer by elementary geometry.

4. Find the length of the curves determined by the following paths. For each path, determine the velocity of the path at $t = 0$.
   (a) The path $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ defined for all $t \in [0, 2\pi]$ as

$$\boldsymbol{\gamma}(t) = \big(\cos t + t \sin t\big)\hat{\boldsymbol{\imath}} + \big(\sin t - t \cos t\big)\hat{\boldsymbol{\jmath}}.$$

   (b) (The original problem had an error, so this problem has been changed from its original).
   The path $\boldsymbol{\gamma} : [0, 3] \to \mathbb{R}^2$ given by coordinates $\boldsymbol{\gamma}(t) = (x(t), y(t))$ that are defined for all $t \in [0, \sqrt{3}]$ as

$$x(t) = 3t - t^3 \qquad \text{and} \qquad y(t) = 3t^2.$$

   (c) The path $\boldsymbol{\gamma} : [0, 1] \to \mathbb{R}^3$ defined for all $t \in [-1, 1]$ as

$$\boldsymbol{\gamma}(t) = \sqrt{2}\sqrt{t+1}(\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - 2\sqrt{1-t}\,\hat{\boldsymbol{\imath}}.$$

5. Find the distance traveled by the particle traveling along the path $\boldsymbol{\gamma} : [0, 2] \to \mathbb{R}^3$ defined for all $t \in [0, 2]$ as

$$\boldsymbol{\gamma}(t) = e^t\hat{\boldsymbol{\imath}} + t\sqrt{2}\hat{\boldsymbol{\jmath}} + e^{-t}\hat{\boldsymbol{k}}.$$

6. A path $\boldsymbol{\gamma}$ is said to be a *unit speed parameterization* of a curve if it parameterizes that curve and has constant speed $\|\boldsymbol{\gamma}'(t)\| = 1$.

(a) Let Γ be a C^1 simple curve and let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a *regular parameterization* (which means that $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$ for all t). A unit speed parameterization of Γ can be found as follows. Define the function $f : [a, b] \to \mathbb{R}$ for all $t \in [a, b]$ as

$$f(t) = \int_a^t \|\boldsymbol{\gamma}'(s)\| \, ds.$$

Note that $f(a) = 0$ and that the length of the curve is $f(b) = L$. For any $t \in [a, b]$, the value of $f(t)$ is the distance traced out by the path $\boldsymbol{\gamma}$ starting at time 0 and ending at time t. From the Fundamental Theorem of Calculus, we have

$$f'(t) = \frac{d}{dt} \int_a^t \|\boldsymbol{\gamma}'(s)\| \, ds = \|\boldsymbol{\gamma}'(t)\|. \tag{1}$$

The assumption that $\boldsymbol{\gamma}$ is regular implies that $\|\boldsymbol{\gamma}'(t)\| > 0$ for all t. Hence f is strictly increasing, as $f'(t) > 0$ for all t, and thus f is invertible.

Problem: Show that the path $\boldsymbol{\beta} : [0, L] \to \mathbb{R}^n$ defined by

$$\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(f^{-1}(t))$$

is a unit speed parameterization by showing that its speed is constant and equal to 1.

(b) Let $\boldsymbol{\gamma} : [0, 5] \to \mathbb{R}^2$ be the path defined by

$$\boldsymbol{\gamma}(t) = (e^t \cos t, e^t \sin t).$$

i. Sketch the curve traced out by this path.
ii. Find the length of the resulting curve.
iii. Find a unit speed parameterization of this curve.