ECE 206 Fall 2019
Practice Problems Week 1
Solutions

1. Consider the vectors $\boldsymbol{u} = (1, -2, 2)$ and $\boldsymbol{v} = (1, 1, -4)$.
   (a) Compute the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.
   (b) Compute the area of the parallelogram defined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution. (a) The lengths of the vectors are

$$\|\boldsymbol{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3 \quad \text{and} \quad \|\boldsymbol{v}\| = \sqrt{1^2 + 1^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2},$$

while their dot product is equal to

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1)(1) + (-2)(1) + (2)(-4) = 1 + 2 + 8 = -9.$$

The angle between the two vectors is therefore

$$\begin{aligned} \theta = \arccos\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right) &= \arccos\left(\frac{-9}{9\sqrt{2}}\right) \\ &= \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}. \end{aligned}$$

(b) The area of the parallelogram is equal to $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$, where θ is the angle between the two vectors. Since $\theta = 3\pi/4$ (as computed above) and $\sin(3\pi/4) = 1/\sqrt{2}$, this area is equal to

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta = 9\sqrt{2} \sin\frac{3\pi}{4} = 9\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 9.$$

Alternatively, one may compute this area as $\|\boldsymbol{u} \times \boldsymbol{v}\|$, where

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & -2 & 2 \\ 1 & 1 & -4 \end{vmatrix} \\ &= ((-2)(-4) - (2)(1))\hat{\boldsymbol{\imath}} + ((2)(1) - (1)(-4))\hat{\boldsymbol{\jmath}} + ((1)(1) - (-2)(1))\hat{\boldsymbol{k}} \\ &= (6, 6, 3) = 3\,(2, 2, 1) \end{aligned}$$

and thus $\|\boldsymbol{u} \times \boldsymbol{v}\| = 3\|(2, 2, 1)\| = 3\sqrt{2^2 + 2^2 + 1^2} = 3\sqrt{9} = 9.$

2. Find a parametric representation of the following curves. (Hint: use the identity $\sin^2\theta + \cos^2\theta = 1$.)

(a) The intersection of the ellipsoid $\{(x, y, z) : x^2 + 3y^2 + 4z^2 = 1\}$ with the plane $\{(x, y, z) : y = x\}$.

(b) The intersection of the elliptical cylinder $\{(x, y, z) : x^2 + 4y^2 = 4\}$ with the hyperboloid $\{(x, y, z) : z^2 = 2 + x + y^2\}$ in the region where $z \geq 0$.

(c) The helix that lies on the cylinder $\{(x, y, z) : x^2 + y^2 = 4\}$ starting at $(2, 0, 0)$ and ending at $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ and going counter-clockwise around the z-axis.

Solution. .

(a) The points on the curve must satisfy both

$$x^2 + 3y^2 + 4z^2 = 1 \qquad \text{and} \qquad x = y.$$

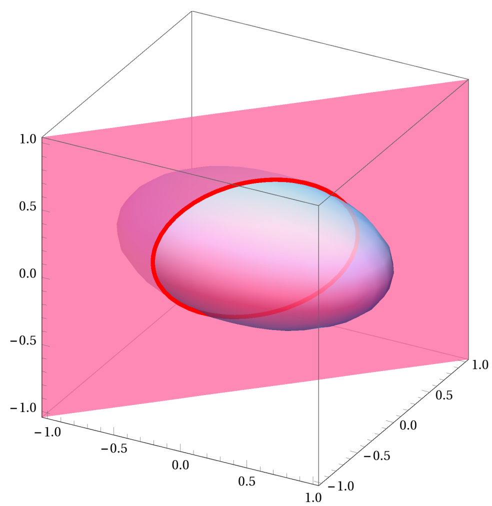
Substituting $x = y$ in the first equation, we see that the points must satisfy

$$4x^2 + 4z^2 = 1,$$

so we may parameterize the x- and z-coordinates as $x(t) = (\sin t)/2$ and $z(t) = (\cos t)/2$. We can define the path $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^3$ by taking $y(t) = x(t)$ as

$$\boldsymbol{\gamma}(t) = \frac{1}{2} (\sin t,\ \sin t,\ \cos t),$$

which parametrizes the curve. A visualization of this curve is provided below.



This is the circle of radius 1/2 on the plane $y = x$ and centered at the origin.

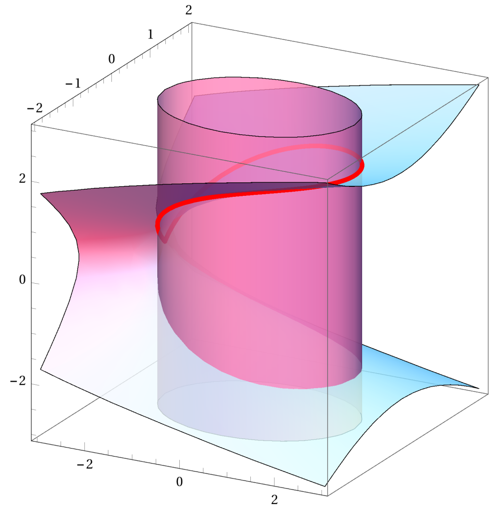
(b) Note that the curve sits on the ellipse $x^2/4 + y^2 = 1$ centered along the z-axis, so we may parameterize the x- and y-coordinates as $x(t) = 2 \sin t$ and $y(t) = \cos(t)$. Solving for z, we have

$$\begin{aligned} z(t)^2 &= 2 + x + y^2 \\ &= 2 + 2\sin t - \cos^2 t \end{aligned}$$

or $z(t) = \sqrt{2 + 2\sin t + \cos^2 t}$ (where we take the positive root, since we are restricted to the region where $z \geq 0$). The desired path is

$$\boldsymbol{\gamma}(t) = \left(2\sin t,\ \cos t,\ \sqrt{2 + 2\sin t + \cos^2 t}\right),$$

for the range $0 \leq t \leq 2\pi$. A visualization of this curve is provided below.

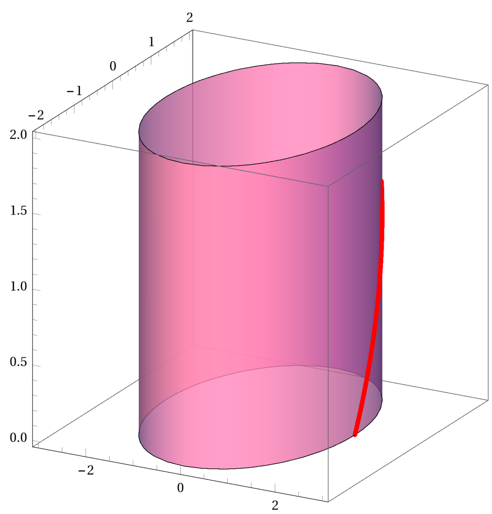


This curve looks like an ellipse when viewed from above.

(c) We may take $x(t) = 2\cos t$ and $y(t) = 2\sin t$. The z-coordinate must be of the form $z(t) = at$ for some constant $a \in \mathbb{R}$. To find a, note that at the time $t = \pi/4$ we have $x(\pi/4) = y(\pi/4) = \sqrt{2}$ and $z(t) = a\pi/4 = \sqrt{2}$. Solving for a, we find $a = 4\sqrt{2}/\pi$. One parameterization is therefore

$$\boldsymbol{\gamma}(t) = \left(2\cos t,\, 2\sin t,\, \frac{4\sqrt{2}}{\pi}t\right).$$

A visualization of this curve is provided below.



3. Let $a \in \mathbb{R}$ be a constant. Give a parametric representation of the circle

$$\Gamma = \{(x, y, z) : x^2 + y^2 + z^2 = 4a^2 \text{ and } x = a\}$$

and find its total length by integration. Check your answer by elementary geometry.

Solution. One may simplify the equations defining the curve to $y^2 + z^2 = 3a^2$ and $x = a$. These equations define the circle on the plane defined by $x = a$ with centre at $(a, 0, 0)$ and radius equatl to $a\sqrt{3}$. One parameterization is

$$\boldsymbol{\gamma}(t) = \left(a,\; a\sqrt{3}\sin t,\; a\sqrt{3}\cos t\right)$$

for $t \in [0, 2\pi]$. The speed of this path is

$$\|\boldsymbol{\gamma}'(t)\| = \sqrt{3a^2\cos^2 t + 3a^2\sin^2 t} = |a|\sqrt{3}$$

and the length of the curve is therefore

$$\int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\|\, dt = \int_0^{2\pi} |a|\sqrt{3}\, dt = 2\pi|a|\sqrt{3},$$

which is 2π times the radius of the circle (exactly what we expect).

4. Find the length of the curves determined by the following paths. For each path, determine the velocity of the path at $t = 0$.

   (a) The path $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ defined for all $t \in [0, 2\pi]$ as

$$\boldsymbol{\gamma}(t) = \left(\cos t + t\sin t\right)\hat{\boldsymbol{\imath}} + \left(\sin t - t\cos t\right)\hat{\boldsymbol{\jmath}}.$$

Solution. The velocity of this path is given by

$$\boldsymbol{\gamma}'(t) = (-\sin t + \sin t + t\cos t)\hat{\boldsymbol{\imath}} + (\cos t - \cos t + t\sin t)\hat{\boldsymbol{\jmath}} = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}}.$$

The velocity at $t = 0$ is therefore $\boldsymbol{\gamma}'(0) = \mathbf{0}$. The speed of the path at any time is

$$\|\boldsymbol{\gamma}'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t,$$

and the length of the curve is therefore

$$\int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\|\, dt = \int_0^{2\pi} t\, dt = \frac{1}{2}t^2\Big|_0^{2\pi} = \frac{4\pi^2}{2} = 2\pi^2.$$

   (b) (The original problem had an error, so this problem has been changed from its original). The path $\boldsymbol{\gamma} : [0, 3] \to \mathbb{R}^2$ given by coordinates $\boldsymbol{\gamma}(t) = (x(t), y(t))$ that are defined for all $t \in [0, \sqrt{3}]$ as

$$x(t) = 3t - t^3 \qquad \text{and} \qquad y(t) = 3t^2.$$

Solution. The velocity of this path is given by

$$\boldsymbol{\gamma}'(t) = \left(3 - 3t^2,\; 6t\right) = 3(1 - t^2, 2t).$$

The velocity at $t = 0$ is therefore $\boldsymbol{\gamma}'(0) = (3, 0)$. The speed of the path at any time is

$$\|\boldsymbol{\gamma}'(t)\| = 3\sqrt{(1 - t^2)^2 + (2t)^2} = 3\sqrt{t^4 + 2t^2 + 1} = 3\sqrt{(t^2 + 1)^2} = 3(t^2 + 1),$$

and the length of the curve is therefore

$$\int_0^{\sqrt{3}} \|\gamma'(t)\|\, dt = 3\int_0^{\sqrt{3}} (t^2+1)\, dt = 3\left(\frac{1}{3}t^3 + t\right)\Big|_0^{\sqrt{3}}$$
$$= 3\left(\frac{1}{3}3\sqrt{3} + \sqrt{3}\right) = 6\sqrt{3}.$$

(c) The path $\gamma : [0,1] \to \mathbb{R}^3$ defined for all $t \in [-1,1]$ as

$$\gamma(t) = \sqrt{2}\sqrt{t+1}(\hat{\jmath} + \hat{k}) - 2\sqrt{1-t}\,\hat{\imath}.$$

Solution. The velocity of this path is given by

$$\gamma'(t) = \left(\frac{1}{\sqrt{1-t}}, \frac{1}{\sqrt{2}\sqrt{t+1}}, \frac{1}{\sqrt{2}\sqrt{t+1}}\right)$$

The velocity at $t = 0$ is therefore $\gamma'(0) = (1, 1/\sqrt{2}, \sqrt{2})$. The speed of the path at any time is

$$\|\gamma'(t)\| = \sqrt{\frac{1}{1-t} + \frac{1}{2}\frac{1}{t+1} + \frac{1}{2}\frac{1}{t+1}} = \sqrt{\frac{1}{1-t} + \frac{1}{1+t}} = \sqrt{\frac{2}{1-t^2}} = \frac{\sqrt{2}}{\sqrt{1-t^2}},$$

and the length of the curve is therefore

$$\int_{-1}^{1} \|\gamma'(t)\|\, dt = \sqrt{2}\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt = \sqrt{2}\arcsin(t)\Big|_{t=-1}^{1}$$
$$= \sqrt{2}\big(\arcsin(1) - \arcsin(-1)\big) = \sqrt{2}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \sqrt{2}\pi.$$

5. Find the distance traveled by the particle traveling along the path $\gamma : [0,2] \to \mathbb{R}^3$ defined for all $t \in [0,2]$ as

$$\gamma(t) = e^t\hat{\imath} + t\sqrt{2}\hat{\jmath} + e^{-t}\hat{k}.$$

Solution. The speed of the particle at any time t is given by

$$\|\gamma'(t)\| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$$

The distance that the particle travels can therefore be computed as

$$\int_0^2 \|\gamma'(t)\|\, dt = \int_0^2 \left(e^t + e^{-t}\right)\, dt = \left[e^t - e^{-t}\right]_{t=0}^2 = e^2 - e^{-2} = e^2 - \frac{1}{e^2}.$$

6. A path γ is said to be a *unit speed parameterization* of a curve if it parameterizes that curve and has constant speed $\|\gamma'(t)\| = 1$.

(a) Let Γ be a C^1 simple curve and let $\gamma : [a,b] \to \mathbb{R}^n$ be a *regular parameterization* (which means that $\gamma'(t) \neq \mathbf{0}$ for all t). A unit speed parameterization of Γ can be found as follows. Define the function $f : [a,b] \to \mathbb{R}$ for all $t \in [a,b]$ as

$$f(t) = \int_a^t \|\boldsymbol{\gamma}'(s)\|\, ds.$$

Note that $f(a) = 0$ and that the length of the curve is $f(b) = L$. For any $t \in [a,b]$, the value of $f(t)$ is the distance traced out by the path $\boldsymbol{\gamma}$ starting at time 0 and ending at time t. From the Fundamental Theorem of Calculus, we have

$$f'(t) = \frac{d}{dt}\int_a^t \|\boldsymbol{\gamma}'(s)\|\, ds = \|\boldsymbol{\gamma}'(t)\|. \tag{1}$$

The assumption that $\boldsymbol{\gamma}$ is regular implies that $\|\boldsymbol{\gamma}'(t)\| > 0$ for all t. Hence f is strictly increasing, as $f'(t) > 0$ for all t, and thus f is invertible.

Problem: Show that the path $\boldsymbol{\beta} : [0, L] \to \mathbb{R}^n$ defined by

$$\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(f^{-1}(t))$$

is a unit speed parameterization by showing that its speed is constant and equal to 1.

Solution. Recall from first semester calculus that the derivative of an inverse function is computed as

$$\frac{d}{dt} f^{-1}(t) = \frac{1}{f'(f^{-1}(t))}.$$

From (1), we see that $f'(t) = \|\boldsymbol{\gamma}'(t)\|$. The velocity of the path $\boldsymbol{\beta}$ is therefore

$$\boldsymbol{\beta}'(t) = \frac{1}{f'(f^{-1}(t))}\boldsymbol{\gamma}'(f^{-1}(t)) = \frac{1}{\|\boldsymbol{\gamma}'(f^{-1}(t))\|}\boldsymbol{\gamma}'(f^{-1}(t))$$

and the speed is

$$\|\boldsymbol{\beta}'(t)\| = \frac{\|\boldsymbol{\gamma}'(f^{-1}(t))\|}{\|\boldsymbol{\gamma}'(f^{-1}(t))\|} = 1.$$

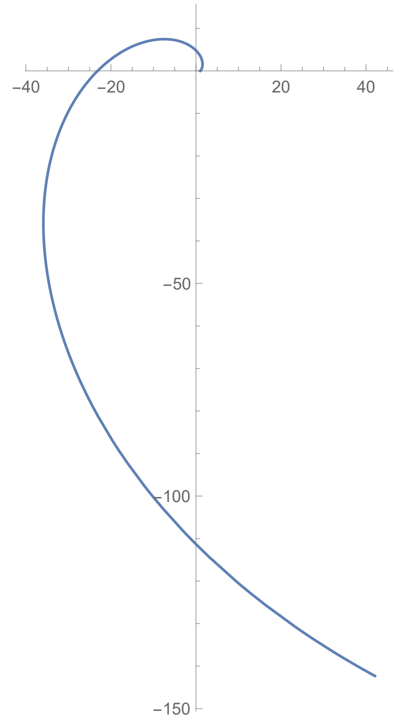
Since the speed of $\boldsymbol{\beta}$ is constant at 1, this path is a unit speed parameterization.

(b) Let $\boldsymbol{\gamma} : [0, 5] \to \mathbb{R}^2$ be the path defined by

$$\boldsymbol{\gamma}(t) = (e^t \cos t, e^t \sin t).$$

i. Sketch the curve traced out by this path.

Solution. The curve traced out is a 'spiral'.



ii. Find the length of the resulting curve.

Solution. The speed of the path is

$$\begin{aligned}\|\gamma'(t)\| &= \big\|(e^t(\cos t - \sin t), e^t(\sin t + \cos t))\big\| \\ &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} \\ &= e^t\sqrt{\left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right) + \left(\sin^2 t + 2\cos t \sin t + \cos^2 t\right)} \\ &= e^t\sqrt{2(\cos^2 t + \sin^2 t)} = \sqrt{2}e^t.\end{aligned}$$

The length of the curve is therefore

$$L = \int_0^5 \|\gamma'(t)\|\, dt = \sqrt{2}\int_0^5 e^t\, dt = \sqrt{2}e^t\Big|_0^5 = \sqrt{2}\left(e^5 - 1\right)$$

iii. Find a unit speed parameterization of this curve.

Solution. To find the unit speed parameterization, define the function

$$f(t) = \int_0^t \|\gamma'(s)\|\, ds = \sqrt{2}\int_0^t e^s\, ds = \sqrt{2}(e^t - 1).$$

The inverse of f is therefore

$$f^{-1}(t) = \ln\frac{t + \sqrt{2}}{\sqrt{2}}$$

and we may define a unit speed parameterization $\boldsymbol{\beta} : [0, L] \to \mathbb{R}^2$ as

$$\begin{aligned}\boldsymbol{\beta}(t) &= \boldsymbol{\gamma}(f^{-1}(t)) \\ &= \frac{t+\sqrt{2}}{\sqrt{2}}\left(\cos\left(\ln\frac{t+\sqrt{2}}{\sqrt{2}}\right), \sin\left(\ln\frac{t+\sqrt{2}}{\sqrt{2}}\right)\right).\end{aligned}$$

This is rather unwieldy, but if we were to take the derivative of this path to find the speed, we would find that $\|\boldsymbol{\beta}'(t)\| = 1$ for all $t \in [0, L]$. So $\boldsymbol{\beta}$ is indeed a unit speed parameterization.