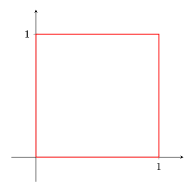
ECE 206 Fall 2019
Practice Problems Week 2

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = 2xy$. Compute the path integral $\int_\Gamma f\, ds$ for Γ to be the following paths.

   (a) Where Γ is the curve traced out by the path $\boldsymbol{\gamma} : [0, 4] \to \mathbb{R}^2$ defined as $\boldsymbol{\gamma}(t) = (t,\, t + 1)$.

   (b) Where Γ is the square whose vertices are (0,0), (0,1), (1,0), and (1,1).



2. Suppose that a wire has the shape of a helix of radius R and height h. Its shape is the curve determined by the path defined by $\boldsymbol{\gamma}(t) = (R \cos t, R \sin t, \frac{h}{2\pi} t)$ for $0 \leq t \leq 2\pi$. Suppose that the linear density of the wire varies linearly with height, so that it can be described by the function $\rho(x, y, z) = \rho_0 \left(1 + (k - 1) \frac{z}{h}\right)$, where k and ρ_0 are constants. Find the mass of the wire.

3. For the following vector fields $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$, find the equation for the flow lines and make a sketch of the field and flow lines.

   (a) $\boldsymbol{F}(x, y) = (x, x^2)$.

   (b) $\boldsymbol{F}(x, y) = (-y, x)$.

   (c) $\boldsymbol{F}(x, y) = (-x, y)$.

4. A static, electrically charged particle located at the origin with positive charge q produces the electric field $\boldsymbol{E} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as
$$\boldsymbol{E}(\boldsymbol{r}) = kq \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)$$
where $\boldsymbol{r} = (x, y, z)$, $r = \|\boldsymbol{r}\|$, and k is a constant. Find the flow lines of $\boldsymbol{E}$ and make a sketch of the field and flow lines.

5. Let $a, b > 0$ be positive constants. For the field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined in 3(b), compute the line integral of $\boldsymbol{F}$ along the ellipse $\{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ oriented counterclockwise.

6. Let Γ be the curve that is the intersection of the cylinder $y^2 + x^2 = 1$ with the plane $z = 3x$, oriented in the counterclockwise direction as viewed from the positive z-axis. Evaluate $\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field given by $\boldsymbol{F}(x, y, z) = (yz, -xz, y)$.

7. A force field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\boldsymbol{F}(x, y, z) = (x, y, z)$. Calculate the work done by $\boldsymbol{F}$ in moving a particle along the parabola $z = 0$, $y = x^2$ from $x = -1$ to $x = 2$.

8. A force field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\boldsymbol{F}(x, y) = (x^3, y, z)$. Calculate the work done by $\boldsymbol{F}$ on a particle moving along the path $\boldsymbol{\gamma}(t) = (0, a\cos t, a\sin t)$ for $t \in [0, 2\pi]$, where $a > 0$ is a constant. Explain the result geometrically.

9. Test whether the following fields are gradient fields. If they are, find the corresponding scalar potential functions.

   (a) $\boldsymbol{F}(x, y) = (3x^2y, x^3)$

   (b) $\boldsymbol{F}(x, y) = (2xe^y + y)\hat{\boldsymbol{\imath}} + (x^2e^y + x - 2y)\hat{\boldsymbol{\jmath}}$

   (c) $\boldsymbol{F}(x, y) = (6 - 2xy + y^3)\hat{\boldsymbol{\imath}} + (x^2 - 8y + 3xy^2)\hat{\boldsymbol{\jmath}}$

   (d) $\boldsymbol{F}(x, y, z) = \big(2xyz^3 + ye^{xy},\ x^2z^3 + xe^{xy},\ 3x^2yz^2 + \cos z\big)$.

10. A force field is given by $\boldsymbol{F}(x, y) = (y, x)$. Compute the work done by $\boldsymbol{F}$ to move a particle along the path $\boldsymbol{\gamma}(t) = \big(t^9,\ \sin^9(\frac{\pi}{2}t)\big)$ for $0 \le t \le 1$. (Hint: the line integral is messy...)

11. Consider the vector field $\boldsymbol{F} = (y^2\cos x + z^3, 2y\sin x - 4, 3xz^2 + z)$. Compute the line integral $\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$, where Γ is any curve in $\mathbb{R}^3$ from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$.

12. The electric field from a point charge at the origin with charge q is $\boldsymbol{E}(\boldsymbol{r}) = \dfrac{kq}{r^3}\boldsymbol{r}$, where $\boldsymbol{r} = (x, y, z)$ and $r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, and k is the electrostatic constant. The Coulomb force that is experienced by another point charge with charge Q at any point $\boldsymbol{r}$ is $\boldsymbol{F}(\boldsymbol{r}) = Q\boldsymbol{E}(\boldsymbol{r})$.

    (a) Show that the Coulomb force field $\boldsymbol{F}$ is conservative by finding a scalar potential function Ψ such that $\boldsymbol{F} = \nabla\Psi$.

    (b) Find an expression for the work done by the field $\boldsymbol{F}$ to move the charge Q from the point $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ to the point $\boldsymbol{r}_2 = (x_2, y_2, z_2)$.
    (You may let $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$ to simplify the final result.)

    (c) When $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the same distance from the origin, what is the physical interpretation of the answer in part (b)?

13. Suppose $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a gradient field. Then there exists a scalar potential $\psi : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{F}(\boldsymbol{r}) = \nabla\psi(\boldsymbol{r})$, where $\boldsymbol{r} = (x, y, z)$. In physics, the *potential energy* of an object is defined as $P(\boldsymbol{r}) = -\psi(\boldsymbol{r})$, so that $\boldsymbol{F} = -\nabla P$. Recall that a *flow line* of the field $\boldsymbol{F}$ is a path $\boldsymbol{\gamma}$ such that $\dfrac{d}{dt}\boldsymbol{\gamma}(t) = \boldsymbol{F}(\boldsymbol{\gamma}(t))$ for each t.

    (a) Let $\boldsymbol{\gamma}(t)$ be a flow line of the vector field $\boldsymbol{F}$. Show that $P(\boldsymbol{\gamma}(t))$ is a decreasing function of t. (This implies that the potential energy of a particle following a flow line decreases.)

    (b) Continuing with the setup from above, if the potential energy of the particle decreases, it must go somewhere else. Show that it goes to kinetic energy. In fact, show that a gradient field conserves energy by showing that "potential energy + kinetic energy = constant" for all time.