ECE 206 Fall 2019
Practice Problems Week 2
Solutions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = 2xy$. Compute the path integral $\displaystyle\int_\Gamma f\, ds$ for Γ to be the following paths.

   (a) Where Γ is the curve traced out by the path $\boldsymbol{\gamma} : [0, 4] \to \mathbb{R}^2$ defined as $\boldsymbol{\gamma}(t) = (t,\, t + 1)$.

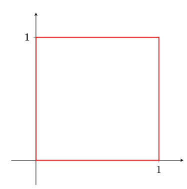
   Solution. The speed of the path at any time t is given by

$$\|\boldsymbol{\gamma}'(t)\| = \|(1, 1)\| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

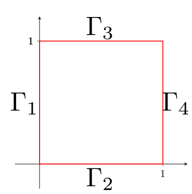
   while the value of f at any point on the path is $f(\boldsymbol{\gamma}(t)) = 2t(t + 1) = 2(t^2 + t)$. The value of the path integral is therefore

$$\begin{aligned}\int_\Gamma f\, ds = \int_0^4 f\big(\boldsymbol{\gamma}(t)\big)\|\boldsymbol{\gamma}'(t)\|\, dt &= 2\sqrt{2}\int_0^4 (t^2 + t)\, dt \\ &= 2\sqrt{2}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_{t=0}^{4} \\ &= 2\sqrt{2}\left(\frac{64}{3} + \frac{16}{2}\right) = \frac{176\sqrt{2}}{3}.\end{aligned}$$

   (b) Where Γ is the square whose vertices are (0,0), (0,1), (1,0), and (1,1).



   Solution. We must split the piecewise C^1 curve into pieces, Γ_1, Γ_2, Γ_3, and Γ_4, which we can do as in the following picture:



where we parameterize each of the pieces

$$\begin{aligned}
&\Gamma_1 \text{ with } \boldsymbol{\gamma}_1(t) = (0,t) \text{ for } 0 \le t \le 1\\
&\Gamma_2 \text{ with } \boldsymbol{\gamma}_2(t) = (t,0) \text{ for } 0 \le t \le 1\\
&\Gamma_3 \text{ with } \boldsymbol{\gamma}_3(t) = (t,1) \text{ for } 0 \le t \le 1\\
&\Gamma_4 \text{ with } \boldsymbol{\gamma}_4(t) = (1,t) \text{ for } 0 \le t \le 1.
\end{aligned}$$

On Γ_1, the value of the scalar field is $f(\boldsymbol{\gamma}_1(t)) = f(0,t) = 0$. Similarly, the scalar field is equal to zero on Γ_2. Thus

$$\int_{\Gamma_1} f\,ds = \int_{\Gamma_2} f\,ds = 0.$$

On Γ_3 and Γ_4, the value of the scalar field is $f(\boldsymbol{\gamma}_3(t)) = f(\boldsymbol{\gamma}_4(t)) = t$, while the speeds of the paths are

$$\|\boldsymbol{\gamma}_3'(t)\| = \|(1,0)\| = 1 \quad \text{and} \quad \|\boldsymbol{\gamma}_4'(t)\| = \|(0,1)\| = 1.$$

The value of the line integrals of f on Γ_3 and Γ_4 are

$$\int_{\Gamma_3} f\,ds = \int_0^1 f(\boldsymbol{\gamma}_3(t))\,\|\boldsymbol{\gamma}_3'(t)\|\,dt = \int_0^1 2t\,dt = 1,$$

and similarly

$$\int_{\Gamma_4} f\,ds = 1.$$

The line integral over the whole curve is therefore

$$\int_{\Gamma} f\,ds = \int_{\Gamma_1} f\,ds + \int_{\Gamma_2} f\,ds + \int_{\Gamma_3} f\,ds + \int_{\Gamma_4} f\,ds = 0+0+1+1 = 2.$$

2. Suppose that a wire has the shape of a helix of radius R and height h. Its shape is the curve determined by the path defined by $\boldsymbol{\gamma}(t) = (R\cos t, R\sin t, \frac{h}{2\pi}t)$ for $0 \le t \le 2\pi$. Suppose that the linear density of the wire varies linearly with height, so that it can be described by the function $\rho(x,y,z) = \rho_0\left(1 + (k-1)\frac{z}{h}\right)$, where k and ρ_0 are constants. Find the mass of the wire.

Solution. The mass of the wire can be determined by

$$M = \int_{\Gamma} \rho\,ds = \int_0^{2\pi} \rho\big(\boldsymbol{\gamma}(t)\big)\,\|\boldsymbol{\gamma}'(t)\|\,dt$$

Note that the velocity and speed of the path are

$$\boldsymbol{\gamma}'(t) = \left(-R\sin t,\, R\cos t,\, \frac{h}{2\pi}\right) \qquad \text{and} \qquad \|\boldsymbol{\gamma}'(t)\| = \sqrt{R^2 + \frac{h^2}{4\pi^2}}.$$

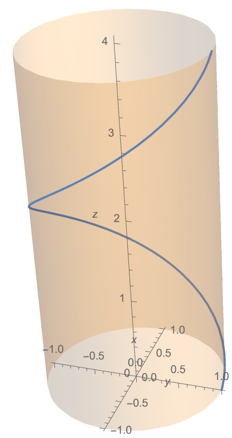
The density of the wire along the path at any point $\boldsymbol{\gamma}(t)$ is given by

$$\rho(\boldsymbol{\gamma}(t)) = \rho_0\left(1 + (k-1)\frac{t}{2\pi}\right).$$

The mass of the wire is therefore

$$M = \rho_0\sqrt{R^2+\frac{h^2}{4\pi^2}}\int_0^{2\pi}\left(1+(k-1)\frac{t}{2\pi}\right)\,dt = \rho_0\sqrt{R^2+\frac{h^2}{4\pi^2}}\left(t+(k-1)\frac{t^2}{4\pi}\right)\Big|_0^{2\pi}$$
$$= \frac{1}{2}(k+1)\rho_0\sqrt{4\pi^2R^2+h^2}$$

after some simplification in the last step. The curve of the wire can be visualized as in the following figure.



3. For the following vector fields $\boldsymbol{F}:\mathbb{R}^2\to\mathbb{R}^2$, find the equation for the flow lines and make a sketch of the field and flow lines.

(a) $\boldsymbol{F}(x,y)=(x,x^2)$.

Solution. By definition, the flow lines are given by paths $\boldsymbol{\gamma}(t)=(x(t),y(t))$ that satisfy the differential equation $\boldsymbol{\gamma}'(t)=\boldsymbol{F}(\boldsymbol{\gamma}(t))$. In component form, this is

$$\frac{dx}{dt}=x \quad\text{and}\quad \frac{dy}{dt}=x^2.$$

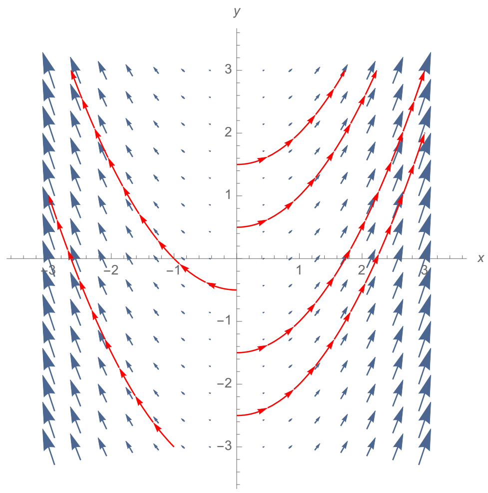
Solving the first differential equation $\frac{dx}{dt}=x$ we find that $x(t)=c_0e^t$ where c_0 is a constant. Plugging this expression for x into the second differential equation we get that

$$\frac{dy}{dt}=c_0^2e^{2t}$$

and solving for y yields $y(t)=\frac{1}{2}c_0^2e^{2t}+c_1$ where c_1 is another constant. It is clear that $y(t)=\frac{1}{2}x(t)^2+c_1$. The flow lines are the curves resulting from the paths

$$\boldsymbol{\gamma}(t)=\left(c_0e^t,\ \frac{1}{2}c_0^2e^{2t}+c_1\right)$$

which may be visualized as in the following figure. The flow lines (red curves) are a family of parabolas that emanate outward from the y-axis. We call the y-axis a **line source**, since the flow lines emanate from it.



Alternatively, to find an expression for the curves, we may simplify the pair of differential equations to $\frac{dy}{dx} = x$, which has solution $y = \frac{x^2}{2} + c$. The flow lines are all (parts of) parabolas with directions as indicated in the figure.

(b) $\boldsymbol{F}(x, y) = (-y, x)$.

Solution. The differential equations for the flow lines are

$$\frac{dx}{dt} = -y \quad \text{and} \quad \frac{dy}{dt} = x.$$

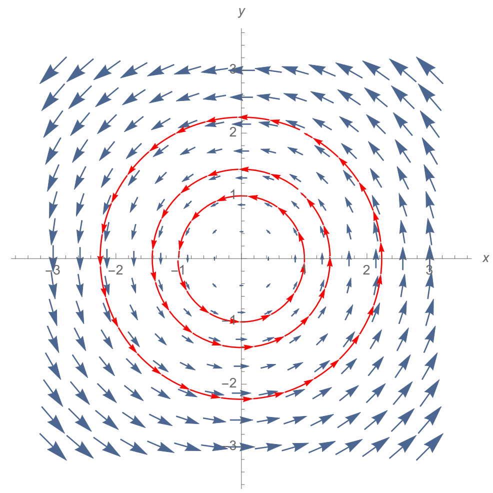
We can simplify the differential equations to obtain

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{or} \qquad y\,dy = -x\,dx.$$

Integrating both sides, we obtain

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \qquad \text{or} \qquad x^2 + y^2 = 2c,$$

which is the equation of a circle with radius $\sqrt{2c}$. Following the direction of the field lines, we see that the paths corresponding to the flow lines circulate counterclockwise around the origin. The field and flow lines can be visualized as in the following figure.



(c) $\boldsymbol{F}(x, y) = (-x, y)$.

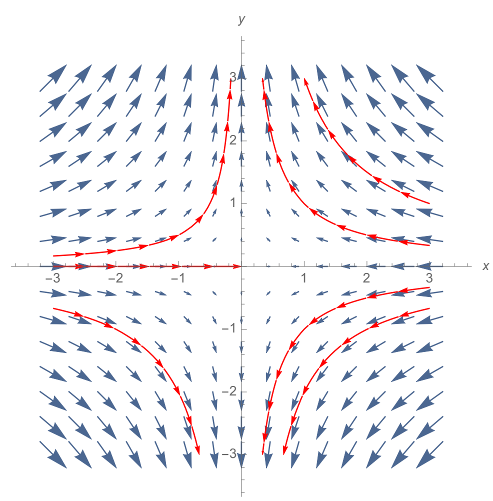
Solution. The differential equations for the flow lines are

$$\frac{dx}{dt} = -x \quad \text{and} \quad \frac{dy}{dt} = y.$$

We can solve each equation separately to find the solutions

$$x(t) = c_0 e^{-t} \qquad \text{and} \qquad y(t) = c_1 e^{t},$$

where c_0 and c_1 are constants. The corresponding curves can be described by the equations $xy = c$ where $C = c_1/c_0$. The field and flow lines can be visualized as in the following figure. These flow lines follow the curves $y = \frac{C}{x}$. Following the field lines, we can see that the paths corresponding to the flow lines have the directions as indicated in the figure.



Alternatively, one may simplify the pair of differential equations above to obtain the differential equation

$$\frac{dy}{dx} = -\frac{y}{x} \quad \text{or} \qquad \frac{dy}{y} = -\frac{dx}{x}.$$

Integrating both sides, one finds that $\ln|y| = -\ln|x| + c_0$ where c_0 is a constant. Simplifying this yields $|y||x| = e^{c_0}$, or $xy = c$, where $c = e^{c_0}$ is another constant.

4. A static, electrically charged particle located at the origin with positive charge q produces the electric field $\boldsymbol{E} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\boldsymbol{E}(\boldsymbol{r}) = kq\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$$

where $\boldsymbol{r} = (x, y, z)$, $r = \|\boldsymbol{r}\|$, and k is a constant. Find the flow lines of $\boldsymbol{E}$ and make a sketch of the field and flow lines.

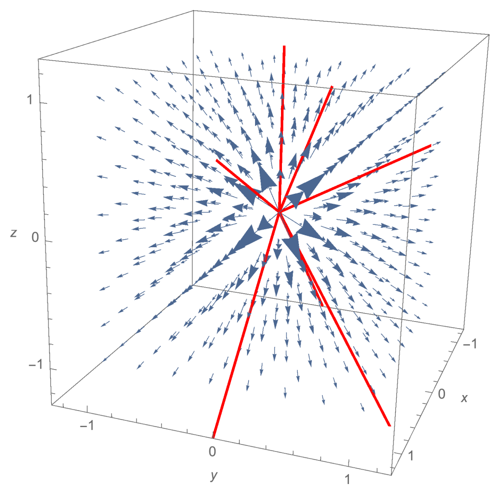
Solution. We have the following system of differential equations:

$$\frac{dx}{dt} = kq\frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{dy}{dt} = kq\frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{dz}{dt} = kq\frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

The idea is to solve the first two equations to get the field lines on the xy-plane, then use the spherical symmetry to rotate them about the x- or y-axes. For now, let $z = 0$. From the first two equations, we get

$$\frac{dy}{dx} = \frac{y}{x}, \Longrightarrow |y| = c|x|, \quad \text{where } c \text{ is an arbitrary constant}$$

These are rays coming from the origin (which is therefore called a **source**). After rotation, we get the field portrait as shown.



Note: if $q < 0$, the field lines would be rays into the origin (a sink).

5. Let $a, b > 0$ be positive constants. For the field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined in 3(b), compute the line integral of $\boldsymbol{F}$ along the ellipse $\{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ oriented counterclockwise.

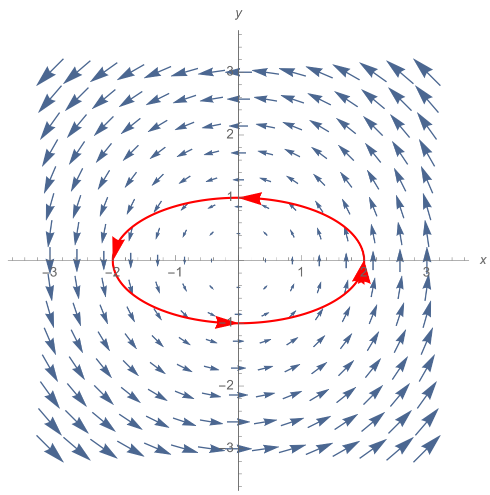
Solution. The vector field is defined as $\boldsymbol{F}(x, y) = (-y, x)$. We can parameterize the curve in the counterclockwise direction as

$$\boldsymbol{\gamma}(t) = \big(a\cos t,\, b\sin t\big)$$

for $t \in [0, 2\pi]$. The velocity of this path at any time t is given by $\boldsymbol{\gamma}'(t) = (-a\sin t, b\cos t)$ and the vector field at any point along the path is equal to $\boldsymbol{F}(\boldsymbol{\gamma}(t)) = (-b\sin t, a\cos t)$. The closed line integral is therefore

$$\begin{aligned}\oint_\Gamma \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt \\ &= \int_0^{2\pi} \big(ab\sin^2 t + ab\cos^2 t\big)\,dt = ab\int_0^{2\pi} dt = 2ab\pi.\end{aligned}$$

The field and curve (with $a = 2$ and $b = 1$) can be visualized as in the following figure.



6. Let Γ be the curve that is the intersection of the cylinder $y^2 + x^2 = 1$ with the plane $z = 3x$, oriented in the counterclockwise direction as viewed from the positive z-axis. Evaluate $\oint_\Gamma \boldsymbol{F}\cdot d\boldsymbol{r}$, where $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field given by $\boldsymbol{F}(x, y, z) = (yz, -xz, y)$.

Solution. We must first parametrize the curve. We may set $x = \cos t$ and $y = \sin t$ and therefore $z = 3\cos t$ to obtain the path

$$\boldsymbol{\gamma}(t) = \big(\cos t,\, \sin t,\, 3\cos t\big)$$

for $t \in [0, 2\pi]$. The velocity of this path is therefore

$$\boldsymbol{\gamma}'(t) = \big(-\sin t,\, \cos t,\, -3\sin t\big)$$

and the vector field at any point along the path is given by

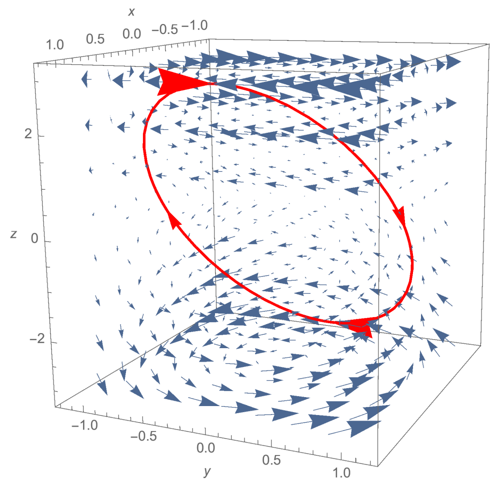
$$\boldsymbol{F}(\boldsymbol{\gamma}(t)) = \big(3\sin t\cos t,\, -3\cos^2 t,\, \sin t\big)\,.$$

The value of the line integral is therefore

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_0^{2\pi} \left(-3\sin^2 t \cos t - 3\cos^3 t - 3\sin^2 t\right) dt$$

$$= -3\int_0^{2\pi} \left(\cos t\left(\sin^2 t + \cos^2 t\right) + \sin^2 t\right) dt$$

$$= -3\int_0^{2\pi} (\cos t + \sin^2 t)\, dt = -3\pi.$$



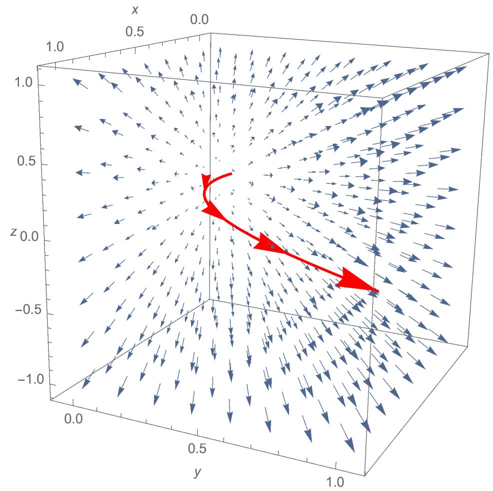
7. A force field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\boldsymbol{F}(x, y, z) = (x, y, z)$. Calculate the work done by $\boldsymbol{F}$ in moving a particle along the parabola $z = 0$, $y = x^2$ from $x = -1$ to $x = 2$.

Solution. The work done is

$$W = \int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$$

where Γ is the parabolic curve. We may parameterize the parabolic curve with the path $\boldsymbol{\gamma}(t) = (t, t^2, 0)$ for $t \in [1, 2]$. Then $\boldsymbol{\gamma}'(t) = (1, 2t, 0)$. The vector field at any point along this path is given by $\boldsymbol{F}(\boldsymbol{\gamma}(t)) = (t, t^2, 0)$. Thus, the work may be computed as

$$W = \int_{-1}^2 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_{-1}^2 (t, t^2, 0) \cdot (1, 2t, 0)\, dt = \int_{-1}^2 (t + 2t^3)\, dt = \left(\frac{t^2}{2} + \frac{t^4}{2}\right)\Bigg|_{-1}^2 = 9$$

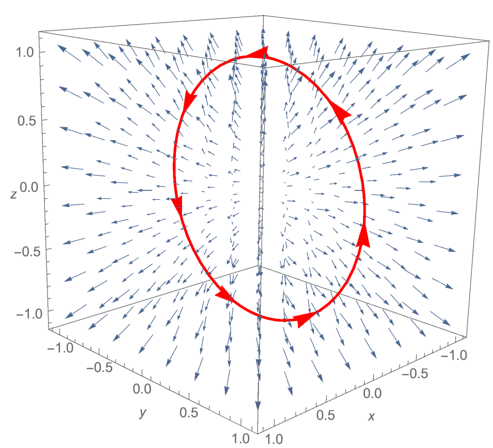


8. A force field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\boldsymbol{F}(x, y) = (x^3, y, z)$. Calculate the work done by $\boldsymbol{F}$ on a particle moving along the path $\boldsymbol{\gamma}(t) = (0, a\cos t, a\sin t)$ for $t \in [0, 2\pi]$, where $a > 0$ is a constant. Explain the result geometrically.

Solution. The work may be computed as

$$W = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_0^{2\pi} (0, a\cos t, a\sin t) \cdot (0, -a\sin t, a\cos t)\, dt$$
$$= \int_0^{2\pi} -a^2 \cos t \sin t + a^2 \cos t \sin t\, dt = 0$$

The path is the circle of radius a centered at (0,0,0) in the yz-plane. In the yz-plane, the vector field components are $(0, y, z)$, so the field lines are rays emanating from the origin. Since the rays emanating from the origin are everywhere orthogonal to this circle, their dot product is zero.



9. Test whether the following fields are gradient fields. If they are, find the corresponding scalar potential functions.

(a) $\boldsymbol{F}(x,y) = (3x^2y, x^3)$

(b) $\boldsymbol{F}(x,y) = (2xe^y + y)\hat{\boldsymbol{\imath}} + (x^2e^y + x - 2y)\hat{\boldsymbol{\jmath}}$

Solution. We first check that

$$\frac{\partial F_2}{\partial x} = 2xe^y + 1 = \frac{\partial F_1}{\partial y}$$

so a potential may exist. Looking at the first component, if a potential function Ψ exists, it must satisfy

$$\frac{\partial \Psi}{\partial x} = 2xe^y + y$$

and thus $\Psi(x,y) = x^2e^y + xy + f(y)$, where $f(y)$ is an arbitrary function that is conatant with respect to x. Differentiating with respect to y and equating to F_2, we find that

$$x^2e^y + x + f'(y) = x^2e^y + x - 2y,$$

which implies that $f'(y) = -2y$ or $f(y) = -y^2 + k$, where k is a constant. Therefore, a scalar potential function for $\boldsymbol{F}$ is

$$\Psi(x,y) = x^2e^y + xy - y^2 + k.$$

(c) $\boldsymbol{F}(x,y) = (6 - 2xy + y^3)\hat{\boldsymbol{\imath}} + (x^2 - 8y + 3xy^2)\hat{\boldsymbol{\jmath}}$

Solution. It can be checked that

$$\frac{\partial F_1}{\partial y} = -2x + 3y^2 \qquad \text{but} \qquad \frac{\partial F_2}{\partial x} = 2x + 3y^2,$$

which are **not** equal (note the difference in minus sign), so the field is not conservative and no potential function exists.

(d) $\boldsymbol{F}(x,y,z) = \big(2xyz^3 + ye^{xy},\, x^2z^3 + xe^{xy},\, 3x^2yz^2 + \cos z\big)$.

Solution. For this vector field, we will see if we can construct a potential function $\Psi : \mathbb{R}^3 \to \mathbb{R}$. If such a potential function exists, it must satisfy

$$\frac{\partial \Psi}{\partial x} = 2xyz^3 + ye^{xy}.$$

Integrating both sides over x, we find that

$$\Psi(x,y,z) = x^2yz^3 + e^{xy} + f(y,z),$$

where f is some function of y and z and is independent of x. Similarly,Ψ must also satisfy

$$\frac{\partial \Psi}{\partial y} = x^2z^3 + xe^{xy}.$$

Differentiating the expression for Ψ above, we find

$$\frac{\partial \Psi}{\partial y} = x^2z^3 + xe^{xy} + \frac{\partial f}{\partial y}(y,z)$$

from which we conclude that $\frac{\partial f}{\partial y}(y,z) = 0$. Therefore f cannot depend on y and it must be a function of z alone, so we may set $f(y,z) = g(z)$ where we need to determine the function g. Lastly, we see that Ψ must satisfy

$$\frac{\partial \Psi}{\partial z} = 3x^2yz^2 + \cos z.$$

Differentiating the expression for Ψ above, we find

$$\frac{\partial \Psi}{\partial z} = 3x^2yz^2 + \frac{dg}{dz}(z)$$

from which we conclude that $\frac{dg}{dz} = \cos z$, or $g(z) = \sin z + c$, where c is some constant. Hence the scalar function

$$\Psi(x,y,z) = x^2yz^3 + e^{xy} + \sin z + c$$

is a valid potential function for $\boldsymbol{F}$ and thus $\boldsymbol{F}$ is conservative.

10. A force field is given by $\boldsymbol{F}(x,y) = (y,x)$. Compute the work done by $\boldsymbol{F}$ to move a particle along the path $\boldsymbol{\gamma}(t) = \left(t^9, \sin^9(\frac{\pi}{2}t)\right)$ for $0 \le t \le 1$. (Hint: the line integral is messy...)

Solution. The hint suggests that because the path is complicated, we probably don't need the path, meaning that the line integral of $\boldsymbol{F}$ is path independent. So let's find a potential function. By inspection, one finds that the potential function $\Psi(x,y) = xy$ will do. Thus

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\boldsymbol{\gamma}(0)}^{\boldsymbol{\gamma}(1)} \nabla\Psi \cdot d\boldsymbol{r} = \Psi(1,1) - \Psi(0,0) = 1,$$

where $\boldsymbol{\gamma}(0) = (0,0)$ and $\boldsymbol{\gamma}(1) = (1,1)$.

11. Consider the vector field $\boldsymbol{F} = (y^2\cos x + z^3, 2y\sin x - 4, 3xz^2 + z)$. Compute the line integral $\displaystyle\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$, where Γ is any curve in $\mathbb{R}^3$ from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$.

Solution. The problem states that the integral should be path independent, so we must find a valid scalar potential function. We assume that $\boldsymbol{F} = \nabla\Psi$ and try to find Ψ by the usual procedure.

We have that $\dfrac{\partial \Psi}{\partial x} = y^2\cos x + z^3$, and integrating with respect to x gives

$$\Psi(x,y,z) = y^2\sin x + xz^3 + f(y,z),$$

where f is an arbitrary function. Differentiating this expression with respect to y, we find that $\dfrac{\partial \Psi}{\partial y} = 2y\sin x + \dfrac{\partial f}{\partial y}$. For consistency, we must have

$$2y\sin x + \frac{\partial f}{\partial y} = 2y\sin x - 4,$$

which implies $f(y,z) = -4y + g(z)$ and thus $\Psi(x,y,z) = y^2\sin x + xz^3 - 4y + g(z)$, where g is some function that depends only on z. Differentiating Ψ with respect to z and comparing to

F_3, we find that

$$3xz^2 + g'(z) = 3xz^2 + z$$

which implies that $g(z) = \frac{z^2}{2} + c$ and thus $\Psi(x, y, z) = y^2 \sin x + xz^3 - 4y + \frac{z^2}{2} + c$, where c is a constant. Thus, we can evaluate the integral using the fundamental theorem for line integrals as

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} \nabla\Psi \cdot d\boldsymbol{r} = \Psi\left(\frac{\pi}{2}, -1, 2\right) - \Psi(0, 1, -1)) = (7 + 4\pi) - \left(-\frac{7}{2}\right) = 4\pi + \frac{21}{2}.$$

12. The electric field from a point charge at the origin with charge q is $\boldsymbol{E}(\boldsymbol{r}) = \frac{kq}{r^3}\boldsymbol{r}$, where $\boldsymbol{r} = (x, y, z)$ and $r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, and k is the electrostatic constant. The Coulomb force that is experienced by another point charge with charge Q at any point $\boldsymbol{r}$ is $\boldsymbol{F}(\boldsymbol{r}) = Q\boldsymbol{E}(\boldsymbol{r})$.

(a) Show that the Coulomb force field $\boldsymbol{F}$ is conservative by finding a scalar potential function Ψ such that $\boldsymbol{F} = \nabla\Psi$.

Solution. We can write the Coulomb force field as

$$\boldsymbol{F}(x, y, z) = \frac{kQr}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z) = (F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)).$$

It can be checked that the scalar field

$$\Psi(x, y, z) = -\frac{kQq}{\sqrt{x^2 + y^2 + z^2}}, \quad \text{or equivalently,} \quad \Psi(\boldsymbol{r}) = -\frac{kQq}{r},$$

is a potential function for $\boldsymbol{F}$. Indeed, note that

$$\frac{\partial\Psi}{\partial x} = -\left(-\frac{1}{2}\right)\frac{kQq}{(x^2 + y^2 + z^2)^{3/2}}2x = \frac{kQrx}{(x^2 + y^2 + z^2)^{3/2}} = F_1(x, y, z).$$

Moreover, it can also be checked that

$$\frac{\partial\Psi}{\partial y} = F_2 \qquad \text{and} \qquad \frac{\partial\Psi}{\partial z} = F_3$$

also hold. Hence Ψ is a potential function for $\boldsymbol{F}$.

(b) Find an expression for the work done by the field $\boldsymbol{F}$ to move the charge Q from the point $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ to the point $\boldsymbol{r}_2 = (x_2, y_2, z_2)$.
(You may let $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$ to simplify the final result.)

Solution. Since the field $\boldsymbol{F}$ is conservative, the work done to move a particle from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is path independent. With the scalar field Ψ defined as above, we have that

$$\Psi(\boldsymbol{r}_1) = -\frac{kQq}{r_1} \quad \text{and } \Psi(\boldsymbol{r}_2) = -\frac{kQq}{r_2},$$

and thus the work done to move the charge Q from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is

$$\text{work} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \nabla\Psi \cdot d\boldsymbol{r} = \Psi(\boldsymbol{r}_2) - \Psi(\boldsymbol{r}_1) = kQq\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

(c) When $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the same distance from the origin, what is the physical interpretation of the answer in part (b)?

Solution. If the initial and final points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the same distance away from the origin (i.e., $r_1 = r_2$), then the work done by the field is 0. The potential energy of the charge Q at $\boldsymbol{r}_1$ is the same as if the particle were at $\boldsymbol{r}_2$. By conservation of energy, the field does not have to do any work to change the location of the particle.

13. Suppose $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a gradient field. Then there exists a scalar potential $\psi : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{F}(\boldsymbol{r}) = \nabla\psi(\boldsymbol{r})$, where $\boldsymbol{r} = (x, y, z)$. In physics, the *potential energy* of an object is defined as $P(\boldsymbol{r}) = -\psi(\boldsymbol{r})$, so that $\boldsymbol{F} = -\nabla P$. Recall that a *flow line* of the field $\boldsymbol{F}$ is a path $\boldsymbol{\gamma}$ such that $\frac{d}{dt}\boldsymbol{\gamma}(t) = \boldsymbol{F}(\boldsymbol{\gamma}(t))$ for each t.

(a) Let $\boldsymbol{\gamma}(t)$ be a flow line of the vector field $\boldsymbol{F}$. Show that $P(\boldsymbol{\gamma}(t))$ is a decreasing function of t. (This implies that the potential energy of a particle following a flow line decreases.)

Solution. Want to show that $\dfrac{dP(\boldsymbol{\gamma}(t))}{dt} < 0$ where $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is a flow line. We must first assume that the path $\boldsymbol{\gamma}$ is not a constant path. Otherwise $\boldsymbol{\gamma}'(t) = \mathbf{0}$ and thus $P(\boldsymbol{\gamma}(t))$ is constant (i.e., $\frac{d}{dt}P(\boldsymbol{\gamma}(t)) = 0$). This implies that $\boldsymbol{F} \neq \mathbf{0}$ on the path, since a flow line must satisfy $\boldsymbol{\gamma}'(t) = \boldsymbol{F}(\boldsymbol{\gamma}(t))$.
Now, at any t, the derivative of the potential energy along the path $\boldsymbol{\gamma}$ is

$$\begin{aligned}
\frac{d}{dt}P(\boldsymbol{\gamma}(t)) &= \frac{d}{dt}P(x(t), y(t), z(t)) \\
&= \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt} \quad \text{(by the chain rule)} \\
&= \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\
&= \nabla P(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \\
&= \nabla P(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{F}(\boldsymbol{\gamma}(t)) \quad \text{(using the equation that defines a field line)} \\
&= \big(-\boldsymbol{F}(\boldsymbol{\gamma}(t))\big) \cdot \boldsymbol{F}(\boldsymbol{\gamma}(t)) \quad \text{(by hypothesis)} \\
&= -\|\boldsymbol{F}(\boldsymbol{\gamma}(t))\|^2 < 0,
\end{aligned}$$

where we have that $\|\boldsymbol{F}(\boldsymbol{\gamma}(t))\| > 0$ since we assumed that $\boldsymbol{\gamma}$ is not constant. This is the desired result.

(b) Continuing with the setup from above, if the potential energy of the particle decreases, it must go somewhere else. Show that it goes to kinetic energy. In fact, show that a gradient field conserves energy by showing that "potential energy + kinetic energy = constant" for all time.

Solution. If $\boldsymbol{F}$ is a gradient field, then the work done along a path $\boldsymbol{\gamma}(t)$ from $t = a$ to $t = b$ is given by

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \nabla P(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = -(P(\boldsymbol{\gamma}(b)) - P(\boldsymbol{\gamma}(a))) = -(P_b - P_a), \qquad (*)$$

where P_a and P_b denote the potential energy of the particle at points $\boldsymbol{\gamma}(a)$ and $\boldsymbol{\gamma}(b)$ respectively.

On the other hand, if m is the mass of the particle and $\boldsymbol{v}(t) = \boldsymbol{\gamma}'(t)$ its velocity, Newton's second law implies that

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d^2\boldsymbol{\gamma}}{dt^2} = m\frac{d}{dt}\left(\frac{d\boldsymbol{\gamma}}{dt}\right) = m\frac{d\boldsymbol{v}}{dt},$$

where $\boldsymbol{a}(t) = \boldsymbol{v}'(t)$ is the acceleration. Another way to compute the work is therefore as follows:

$$\begin{aligned}
\text{work} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,dt \\
&= \int_a^b m\frac{d\boldsymbol{v}(t)}{dt} \cdot \boldsymbol{v}(t)\,dt \\
&= \frac{m}{2}\int_a^b \frac{d}{dt}\big(\boldsymbol{v}(t) \cdot \boldsymbol{v}(t)\big)\,dt \quad \text{(using the product rule in reverse)} \\
&= \frac{m}{2}(\boldsymbol{v}(t) \cdot \boldsymbol{v}(t))\Big|_{t=a}^{b} \quad \text{(by fundamental theorem of calculus)} \\
&= \frac{1}{2}mv(t)^2\Big|_a^b \\
&= \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2 \\
&= K_b - K_a \qquad (**)
\end{aligned}$$

where $K = \frac{1}{2}mv^2$ is the kinetic energy and $v(t) = \|\boldsymbol{v}(t)\|$ is the speed of the particle at time t. Thus, equating $(*)$ with $(**)$ gives:

$$P_a + K_a = P_b + K_b$$

Since a and b are arbitrary times, we have shown that the sum of the potential and kinetic energy is constant at all times (in a gradient field). This is known as *The Conservation of Energy Theorem.*