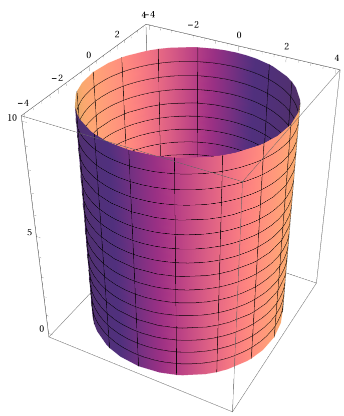
ECE 206 Fall 2019
Practice Problems Week 4
Solutions

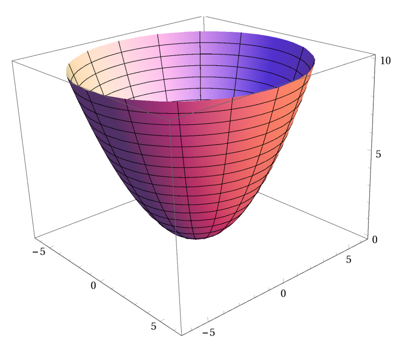
1. Find parametric representations for and sketch each of the following surfaces. (Hint: make use of the identity $\sin^2\theta + \cos^2\theta = 1$.)

   (a) The cylinder $\{(x, y, z) : x^2 + y^2 = a\}$, where $a > 0$ is a constant.

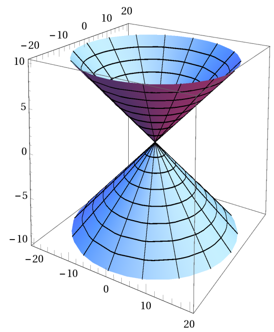
   Solution. One parameterization is $\Phi(s, t) = (\sqrt{a}\sin s, \sqrt{a}\cos s, t)$ for $s \in [0, 2\pi]$ and $t \in \mathbb{R}$. A visualization is provided below.



   (b) The parabaloid $\{(x, y, z) \mid x^2 + y^2 = az\}$, where $a > 0$ is a constant.
   Solution. One parameterization is $\Phi(s, t) = (\sqrt{a}t\sin s, \sqrt{a}t\cos s, t^2)$ for $s \in [0, 2\pi]$ and $t \geq 0$. This can be found by letting $x = \sqrt{a}t\sin s$ and $y = \sqrt{a}t\cos s$. Then $x^2 + y^2 = at^2$ and thus $z = t^2$. A visualization is provided below.



   (c) The cone $\{(x, y, z) \mid x^2 + y^2 = az^2\}$, where $a > 0$ is a constant.
   Solution. One parameterization is $\Phi(s, t) = (\sqrt{a}t\sin s, \sqrt{a}t\cos s, t)$ for $s \in [0, 2\pi]$ and $t \in \mathbb{R}$. This can be found by letting $x = \sqrt{a}t\sin s$ and $y = \sqrt{a}t\cos s$. Then $x^2 + y^2 = at^2 = az^2$ and thus $z = t$. A visualization is provided below.



2. Let $a, b \in \mathbb{R}$ be positive constants and let $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ be the vector-valued function defined as $\boldsymbol{\Phi}(s,t) = (s, a\cos t, b\sin t)$. Consider the surface defined parametrically as

$$\Sigma = \{\boldsymbol{\Phi}(s,t) \,|\, s \in [0,1] \text{ and } t \in [0,\pi]\}.$$

   (a) Determine the normal vector to the surface at any point.

   (b) Determine the equation of the tangent plane to the surface at $(1, 0, b)$.

   (c) Describe and sketch the grid curves $\boldsymbol{\Phi}(0,t)$ and $\boldsymbol{\Phi}(1,t)$ over the range $0 \le v \le \pi$, and the grid curves for $\boldsymbol{\Phi}(s,0)$ and $\boldsymbol{\Phi}(s,\pi)$ over the range $0 \le s \le 1$.

   (d) Set up (but do not compute) up an integral that would compute the area of this surface.

   (e) Sketch the surface.

Solution. .

   (a) To determine the normal plane, we compute the partial derivatives

$$\left.\frac{\partial \boldsymbol{\Phi}}{\partial s}\right|_{(s,t)} = (1,0,0) \qquad \text{and} \qquad \left.\frac{\partial \boldsymbol{\Phi}}{\partial t}\right|_{(s,t)} = (0, -a\sin t, b\cos t).$$

   The normal vector at the point $\boldsymbol{\Phi}(s,t)$ is given by

$$\boldsymbol{n}_{\boldsymbol{\Phi}}(s,t) = \left.\left(\frac{\partial \boldsymbol{\Phi}}{\partial s} \times \frac{\partial \boldsymbol{\Phi}}{\partial t}\right)\right|_{(s,t)} = -\big(0, b\cos t, a\sin t\big).$$

   (b) The point $(1,0,b)$ is given by $\boldsymbol{\Phi}(1,\pi/2)$, and the corresponding normal vector is $\boldsymbol{n}_{\boldsymbol{\Phi}}(1,\pi/2) = (0,0,-a)$. The equation of the plane tangent to the surface at this point is given by $(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0$, or

$$\Big((x,y,z) - (1,0,b)\Big) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}\left(1, \frac{\pi}{2}\right) = 0.$$

   Since $(1,0,b) \cdot (0,0,-a) = -ba$ and $(x,y,z) \cdot (0,0,-a) = -za$, the equation of the plane is therefore $-za = -ba$, or $z = b$ assuming that $a \neq 0$. The plane tangent to the surface is therefore given by

$$\{(z,y,z) \,|\, z = b\}.$$

   (c) The grid curves are:

   - The grid curve defined by $\boldsymbol{\Phi}(0,t) = (0, a\cos t, b\sin t)$ is the curve of the uper half of the ellipse $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ on the plane $x = 0$.
   - The grid curve defined by $\boldsymbol{\Phi}(1,t) = (1, a\cos t, b\sin t)$ is the curve of the uper half of the ellipse $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ on the plane $x = 1$.
   - The grid curve defined by $\boldsymbol{\Phi}(s,0) = (s,1,0)$ is the line segment connecting $(0,1,0)$ to $(1,1,0)$.

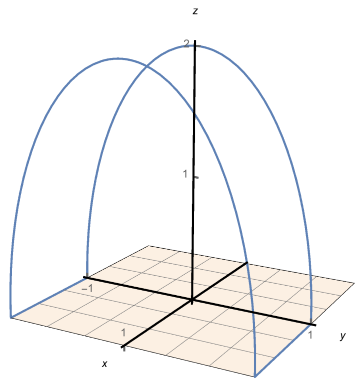


Figure 1: Grid curves.

- The grid curve defined by $\mathbf{\Phi}(s,\pi) = (s,-1,0)$ is the line segment connecting $(0,-1,0)$ to $(1,-1,0)$.

A plot of the grid curves (for $a = 1$ and $b = 2$) can be found in Figure 1.

(d) The magnitude of the normal vector at each point that is determined by $\mathbf{\Phi}$ is

$$\|\boldsymbol{n}_{\mathbf{\Phi}}(s,t)\| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t},$$

so the area of this surface would be equal to the value of the integral

$$\int_0^1 \int_0^\pi \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt\, ds = \int_0^\pi \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$$

which does not have a closed-form expression.

(e) A sketch of the surface (which consists of half of the elliptic cylinder) can be found in Figure **??**.

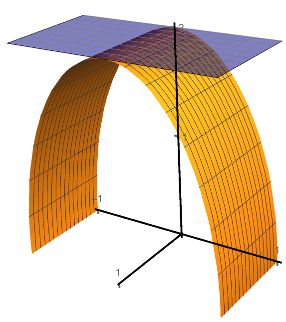


Figure 2: The surface with the tangent plane at the point $(1, 0, b)$.

3. In this problem you will examine the "pringle", which is the surface parameterized by the vector-valued function

$$\mathbf{\Phi}(r,\theta) = \left(r\sin\theta,\ 2r\cos\theta,\ \frac{r^2}{4}\cos 2\theta \right)$$

for $r \in [0, 2]$ and $\theta \in [0, 2\pi]$.

(a) Use a mathematical graphing software to plot the surface.

(b) Determine the normal vector and the equation of the tangent plane to the surface at $(1/\sqrt{2}, \sqrt{2}, 0)$.

(c) Set up (but do not evaluate) an integral to compute the area of the pringle.

Solution. .

(a) Using Mathematica, we may plot the pringle using the following command:

```
ParametricPlot3D[{r Sin[u], 2 r Cos[u], r^2/4 Cos[2 u]},
                                         {r,0, 2}, {u,0,2 Pi}]
```

The result is the graphic provided in Figure 3.

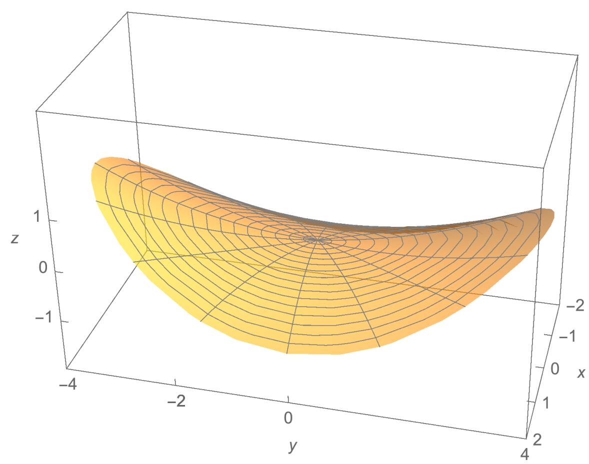


Figure 3: Parametric plot of the pringle.

(b) To determine the normal vector, we must compute the partial derivatives of the parameterization with respect to the parameter variables. We have

$$\left.\frac{\partial \mathbf{\Phi}}{\partial r}\right|_{(r,\theta)} = \left(\sin\theta, 2\cos\theta, \frac{r}{2}\cos 2\theta\right)$$

$$\text{and} \quad \left.\frac{\partial \mathbf{\Phi}}{\partial \theta}\right|_{(r,\theta)} = \left(r\cos\theta, -2r\sin\theta, -\frac{r^2}{2}\sin 2\theta\right).$$

The normal vector at any point will be given by

$$\begin{aligned} \boldsymbol{n}_{\mathbf{\Phi}}(r,\theta) = \left.\left(\frac{\partial \mathbf{\Phi}}{\partial r} \times \frac{\partial \mathbf{\Phi}}{\partial \theta}\right)\right|_{(r,\theta)} &= \left(-r^2\cos\theta\sin 2\theta + r^2\sin\theta\cos 2\theta\right)\hat{\boldsymbol{\imath}} \\ &\quad + \left(\frac{r^2}{2}\sin\theta\sin 2\theta + \frac{r^2}{2}\cos\theta\cos 2\theta\right)\hat{\boldsymbol{\jmath}} - \left(2r\sin^2\theta 2r\cos^2\theta\right)\hat{\boldsymbol{k}} \\ &= -r^2\sin\theta\,\hat{\boldsymbol{\imath}} + \frac{r^2}{2}\cos\theta\,\hat{\boldsymbol{\jmath}} - 2r\,\hat{\boldsymbol{k}}, \end{aligned}$$

where the simplification in the final line above is due to the trigonometric identities

$$\cos\theta\sin 2\theta - \sin\theta\cos 2\theta = \sin\theta \qquad \text{and} \qquad \sin\theta\sin 2\theta + \cos\theta\cos 2\theta = \cos\theta.$$

The point $(1/\sqrt{2}, \sqrt{2}, 0)$ is given by $\boldsymbol{\Phi}(1, \pi/4)$. Evaluating the normal vector at the point defined by $r = 1$ and $\theta = \pi/4$ yields

$$\boldsymbol{n}_{\boldsymbol{\Phi}}\left(1, \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{2\sqrt{2}}\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}.$$

The equation for the plane tangent to the surface at the point $\boldsymbol{r} = \boldsymbol{\Phi}(1, \pi/4) = (1/\sqrt{2}, \sqrt{2}, 0)$ is given by

$$\left\{(x, y, z) \,\middle|\, \Big((x, y, z) - \boldsymbol{r}\Big) \cdot \boldsymbol{n}_{\boldsymbol{\Phi}}(1, \pi/4) = 0\right\}.$$

Note that $\boldsymbol{r} \cdot \boldsymbol{n}_{\boldsymbol{\Phi}} = \frac{1}{2} - \frac{1}{2} = 0$, and thus the plane is described as

$$\left\{(x, y, z) \,\middle|\, -\frac{1}{\sqrt{2}}x + \frac{1}{2\sqrt{2}}y - 2z = 0\right\}.$$

The equation determining the tangent plane is therefore $-\frac{1}{\sqrt{2}}x + \frac{1}{2\sqrt{2}}y - 2z = 0$. A graphic containing the tangent plane to the pringle at this point can be found in Figure 4

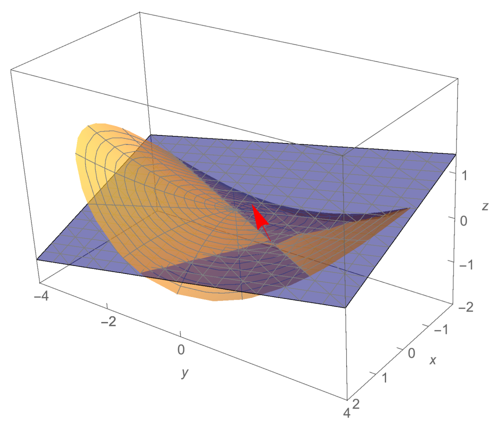


Figure 4: The tangent plane to the pringle at the point $\boldsymbol{r} = \boldsymbol{\Phi}(1, \pi/4) = (1/\sqrt{2}, \sqrt{2}, 0)$. The (normalized) normal vector at that point is also indicated.

(c) To compute the area of the pringle, we need to compute the norm of the nomal vector at each point. This is

$$\|\boldsymbol{n}_{\boldsymbol{\Phi}}(r, \theta)\| = \sqrt{r^4\sin^2\theta + \frac{r^4}{4}\cos^2\theta + 4r^2} = \frac{r}{2}\sqrt{16 + r^2(\cos^2\theta + 4\sin^2\theta)}$$

and the area of the pringle is therefore

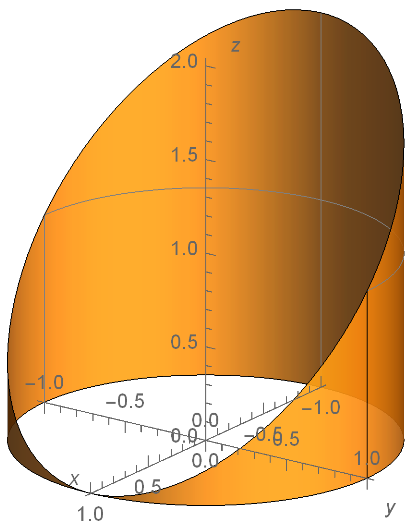
$$\int_0^{2\pi}\int_0^2 \|\boldsymbol{n}_{\boldsymbol{\Phi}}(r, \theta)\|\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r\sqrt{16 + r^2(\cos^2\theta + 4\sin^2\theta)}\, dr\, d\theta,$$

which does not have a closed-form expression!

Nonetheless, we can use a computer to find a numerical estimate. Using Mathematica we can compute this area to be ≈ 28.6821.

4. Determine the area of the following surfaces.

(a) The part of the cylinder defined by $x^2 + y^2 = 1$ and $0 \leq z \leq 1 - x$.
Solution. This surface can be visualized as in the following figure.



One parameterization for this surface is $\mathbf{\Phi}(\theta, r) = (\sin\theta, \cos\theta, r)$ over the region determined by $0 \leq r \leq 1 - \sin\theta$ for all $\theta \in [0, 2\pi]$. We have

$$\left.\frac{\partial \mathbf{\Phi}}{\partial \theta}\right|_{(r,\theta)} = (\cos\theta,\, -\sin\theta,\, 0) \qquad \text{and} \qquad \left.\frac{\partial \mathbf{\Phi}}{\partial r}\right|_{(r,\theta)} = (0,\, 0,\, 1)$$

and the norm of corresponding normal vector is computed as

$$\|\boldsymbol{n}_{\mathbf{\Phi}}(r,\theta)\| = \left\| \left.\left(\frac{\partial \mathbf{\Phi}}{\partial \theta} \times \frac{\partial \mathbf{\Phi}}{\partial r} \right)\right|_{(r,\theta)} \right\| = \|(-\sin\theta,\, -\cos\theta,\, 0)\| = 1.$$

The area is therefore

$$\int_0^{2\pi} \int_0^{1-\sin\theta} dr\, d\theta = \int_0^{2\pi} (1 - \sin\theta)\, d\theta = 2\pi.$$

(b) The paraboloid $\Sigma = \{(x, y, z) \,|\, z = x^2 + y^2 \text{ and } 0 \leq z \leq h\}$, where $h > 0$ is a constant.
Solution. We must first find a suitable parametrization. If we choose $x(r, \theta) = r\sin\theta$ and $y(r, \theta) = r\cos\theta$, we have $z = x^2 + y^2 = r^2$. We may therefore parametrize this surface by

$$\mathbf{\Phi}(r, \theta) = \left(r\sin\theta,\, r\cos\theta,\, r^2\right)$$

over the region[1] defined by $0 \leq r \leq \sqrt{h}$ and $\theta \in [0, 2\pi]$. We have

$$\frac{\partial \mathbf{\Phi}}{\partial r} = (\sin\theta,\, \cos\theta,\, 2r) \qquad \text{and} \qquad \frac{\partial \mathbf{\Phi}}{\partial \theta} = (r\cos\theta,\, -r\sin\theta,\, 0)$$

[1]Note that $z = r^2$ goes from 0 to h, so the paramater r should go from 0 to $\sqrt{h}$.

and the norm of corresponding normal vector is computed as

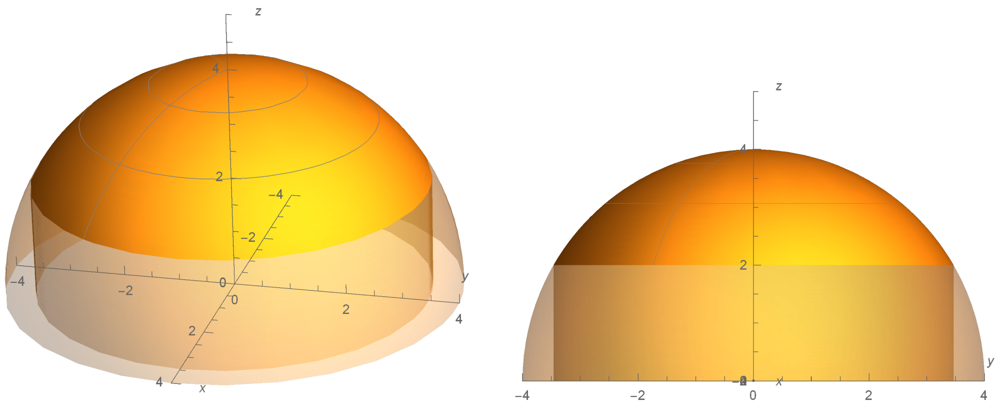
$$\|\boldsymbol{n}_{\boldsymbol{\Phi}}(r,\theta)\| = \left\| \left(\frac{\partial \boldsymbol{\Phi}}{\partial \theta} \times \frac{\partial \boldsymbol{\Phi}}{\partial r} \right)\Big|_{(r,\theta)} \right\| = \left\|(2r^2 \sin\theta,\, 2r^2\cos\theta,\, r)\right\| = r\sqrt{4r^2+1}.$$

The desired area is therefore

$$\int_0^{2\pi} \int_0^{\sqrt{h}} r\sqrt{4r^2+1}\,dr\,d\theta = 2\pi \left[\frac{1}{12}\left(4r^2+1\right)^{3/2} \right]_{r=0}^{\sqrt{h}} = \frac{\pi}{6}\left((4h+1)^{3/2} - 1 \right).$$

(c) The portion of the sphere of radius 4 centered at the origin that lies inside the cylinder determined by $x^2+y^2=12$ and above the xy-plane.

Solution. This surface can be visualized as in the following figure. It is part of the sphere that is inside the cylinder of radius $\sqrt{12}=2\sqrt{3}$.



This portion of the sphere can be parameterized using the standard parametrization for the sphere:

$$\boldsymbol{\Phi}(\varphi,\theta) = \big(4\sin\varphi\cos\theta,\, 4\sin\varphi\sin\theta,\, 4\cos\varphi\big)$$

for $\theta \in [0, 2\pi]$, but we must determine the range for φ. This parameter clearly varies from $\varphi = 0$ to $\varphi = \sin^{-1}\frac{\sqrt{12}}{4} = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$, which the angle that the point of intersection of the cylinder and sphere makes with the z-axis. We have

$$\frac{\partial \boldsymbol{\Phi}}{\partial \varphi} = (4\cos\varphi\cos\theta,\, 4\cos\varphi\sin\theta,\, -4\sin\varphi)$$
$$\text{and} \quad \frac{\partial \boldsymbol{\Phi}}{\partial \theta} = (-4\sin\varphi\sin\theta,\, 4\sin\varphi\cos\theta,\, 0).$$

The corresponding normal vector is

$$\begin{aligned}\boldsymbol{n}_{\boldsymbol{\Phi}}(\theta,\varphi) = \frac{\partial \boldsymbol{\Phi}}{\partial \varphi} \times \frac{\partial \boldsymbol{\Phi}}{\partial \theta} &= \big(16\sin^2\varphi\cos\theta,\, -16\sin^2\varphi\sin\theta,\, 16\cos\varphi\sin\varphi(\sin^2\theta+\cos^2\theta)\big) \\ &= 16\big(\sin^2\varphi\cos\theta,\, -\sin^2\varphi\sin\theta,\, \cos\varphi\sin\varphi\big)\end{aligned}$$

and its norm is computed as

$$\|\boldsymbol{n}_{\boldsymbol{\Phi}}(\theta,\varphi)\| = 16\sqrt{\sin^4\varphi(\cos^2\theta+\sin^2\theta)+\cos^2\varphi\sin^2\varphi} = 16\sin\varphi.$$

The desired area is therefore

$$\int_0^{2\pi}\left(\int_0^{\pi/3} 16\sin\varphi\,d\varphi\right)d\theta = 16(2\pi)(-\cos\varphi)\Big|_{\varphi=0}^{\pi/3} = 32\pi\left(0-\left(-\frac{1}{2}\right)\right) = 16\pi.$$

(d) The 'helicoid' that is parametrized by $\mathbf{\Phi}(r,\theta) = (r\sin\theta,\, r\cos\theta,\, \theta)$ over the domain defined by $r \in [0,1]$ and $\theta \in [0, 4\pi]$. Use the fact that

$$\int \sqrt{1+x^2}\,dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right| + c.$$

Solution. For this parametrization, we have

$$\left.\frac{\partial \mathbf{\Phi}}{\partial r}\right|_{(r,\theta)} = (\sin\theta,\, \cos\theta,\, 0) \qquad \text{and} \qquad \left.\frac{\partial \mathbf{\Phi}}{\partial \theta}\right|_{(r,\theta)} = (r\cos\theta,\, -r\sin\theta,\, 1)$$

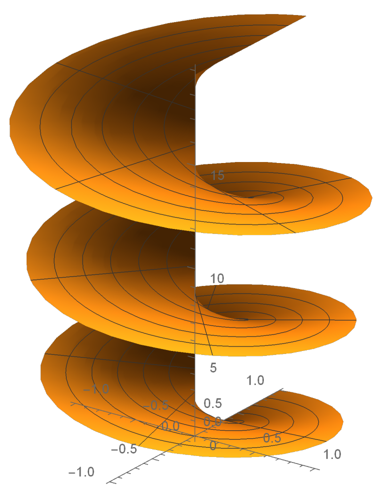
and the norm of corresponding normal vector is computed as

$$\|\boldsymbol{n}_{\mathbf{\Phi}}(r,\theta)\| = \left\| \left.\left(\frac{\partial \mathbf{\Phi}}{\partial r} \times \frac{\partial \mathbf{\Phi}}{\partial \theta}\right)\right|_{(r,\theta)} \right\| = \|(\cos\theta,\, \sin\theta,\, r)\| = \sqrt{r^2+1}.$$

The desired area is therefore

$$\int_0^{4\pi}\int_0^1 \sqrt{r^2+1}\,dr\,d\theta = 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right).$$

This surface looks like this:



5. Evaluate the following surface integrals $\iint_\Sigma f\,dA$ for the following scalar fields and surfaces.

(a) $f(x,y,z) = x/z$ and Σ is the surface parameterized by $\mathbf{\Phi} : D \to \mathbb{R}^3$ defined as

$$\mathbf{\Phi}(s,t) = (t\sin s,\, 1-t^2,\, t\cos s)$$

over the region $D = \{(s,t) \,|\, s \in [0,\pi/3] \text{ and } t \in [0,1]\}$.
Solution. We have

$$\frac{\partial \mathbf{\Phi}}{\partial s} = (t\cos s,\, 0,\, -t\sin s) \qquad \text{and} \qquad \frac{\partial \mathbf{\Phi}}{\partial t} = (\sin s,\, 2t,\, \cos s)$$

and the corresponding normal vector to the surface at any point is

$$\boldsymbol{n}_{\mathbf{\Phi}}(s,t) = \left(2t^2\sin s, t(\cos^2 s + \sin^2 s), 2t^2\cos s\right) = \left(2t^2\sin s, t, 2t^2\cos s\right)$$

and the norm of this vector is

$$\|\boldsymbol{n}_{\boldsymbol{\Phi}}(s,t)\| = \sqrt{4t^4(\cos^2 s + \sin^2 s) + t^2} = \sqrt{4t^4 + t^2} = t\sqrt{4t^2+1}.$$

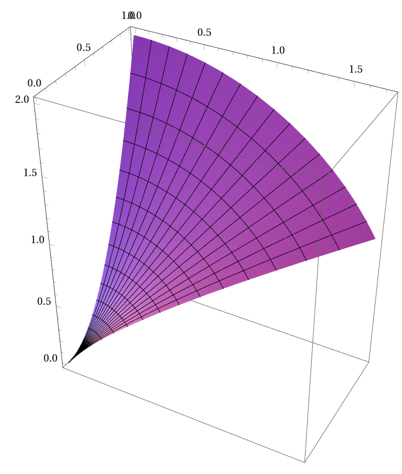
The desired integral is therefore

$$\begin{aligned}
\iint_\Sigma \frac{x}{z}\, dA &= \int_0^{\frac{\pi}{3}} \int_0^1 \frac{t\sin s}{t\cos s} t\sqrt{4t^2+1}\, dt\, ds \\
&= \int_0^{\frac{\pi}{3}} \int_0^1 t\tan s\, \sqrt{4t^2+1}\, dt\, ds \\
&= \left(\int_0^{\frac{\pi}{3}} \tan s\, ds\right) \left(\int_0^1 t\sqrt{4t^2+1}\, dt\right) \\
&= \Big[\ln|\sec s|\Big]_{s=0}^{\frac{\pi}{3}} \left[\frac{1}{12}\left(4t^2+1\right)^{3/2}\right]_{t=0}^1 \\
&= \left(\ln\frac{1}{\cos\frac{\pi}{3}} - \ln\frac{1}{\cos 0}\right) \frac{1}{12}\left(5^{\frac{3}{2}} - 1\right) \\
&= (\ln 2 - \ln 1)\, \frac{5\sqrt{5}-1}{12} \\
&= \frac{5\sqrt{5}-1}{12} \ln 2,
\end{aligned}$$

where the integral $\int t\sqrt{4t^2+1}\, ds$ can be computed using a substitution $u = 4t^2+1$ where $du = 8t\, dt$ such that

$$\int t\sqrt{4t^2+1} dt = \frac{1}{8}\int \sqrt{u}\, du = \frac{1}{8}\frac{2}{3}u^{3/2} + c = \frac{1}{12}(4t^2+1)^{3/2} + c.$$

A visualization of this surface is provided below.



(b) $f(x,y,z) = \sqrt{x^2+y^2+1}$ and Σ is the helicoid surface in problem 4d above.
Solution. We may use the same parametrization and normal vector as above. We have $\|\boldsymbol{n}_{\boldsymbol{\Phi}}(r,\theta)\| = \sqrt{r^2+1}$ with the parameterization $x = r\sin\theta$, $y = r\cos\theta$, and $z = \theta$. The desired integral is

therefore

$$\begin{aligned}\iint_\Sigma \sqrt{x^2+y^2+1}\,dA &= \int_0^{4\pi}\int_0^1 \sqrt{r^2+1}\|\boldsymbol{n}_{\boldsymbol{\Phi}}(r,\theta)\|\,dr\,d\theta = \int_0^{4\pi}\int_0^1 \sqrt{r^2+1}\sqrt{r^2+1}\,dr\,d\theta \\ &= \int_0^{4\pi}\int_0^1 |r^2+1|\,dr\,d\theta \\ &= 4\pi\left(\frac{1}{3}r^3+r\right)\Big|_{r=0}^1 = 4\pi\left(\frac{1}{3}+1\right) = \frac{16\pi}{3}.\end{aligned}$$