ECE 206 Fall 2019
Practice Problems Week 5

1. Use a coordinate transformation to evaluate the following integrals. Make sure to sketch the region of integration in each case.

   (a) The integral $\iint_D (x+y)\,dx\,dy$, where D is the trapezoidal region with vertices given by $(0,0)$, $(1,1)$, $(-4,3)$, and $(-5,2)$.
   (Hint: use the coordinate transformations $x(u,v) = -5u + v$ and $y(u,v) = 2u + v$.)

   (b) Find the volume of the solid under the paraboloid defined by $z = 2 - x^2 - y^2$, above the xy-plane, and inside the cylinder defined by $x^2 + y^2 = 1$.

   (c) The integral $\iint_D x^2\,dx\,dy$, where D is the region inside the ellipse $10x^2 + 6xy + y^2 = 2$.
   Use the coordinate transformations $x(u,v) = \sqrt{2}u$ and $y(u,v) = \sqrt{2}(v - 3u)$ to and verify that this transforms the unit circle in the uv-plane to the region D in the xy-plane. Then use polar coordinates compute the resulting integral.)

2. Let Γ be the closed curve consisting of the semi-circle $x^2 + y^2 = 9$ (with $y \geq 0$) and the x-axis from -3 to 3, oriented in the clockwise direction, and let the vector field $\boldsymbol{F}$ be defined by $\boldsymbol{F} = (x^2y, -xy^2)$.

   Compute $\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$. (Hint: Use Green's theorem, then an appropriate change of coordinates.)

3. Suppose a vector field $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ represents the velocity of the flow of some fluid moving through space with mass density given by the scalar field $\rho : \mathbb{R}^3 \to \mathbb{R}$ (in units of kg per meters squared). The total *mass flux* of the fluid flow is the vector field $\boldsymbol{F}(\boldsymbol{r}) = \rho(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r})$, and the *total mass flux* through a surface Σ is the surface integral $\iint_\Sigma \rho\boldsymbol{v} \cdot d\boldsymbol{A}$.

   Let $b, \ell > 0$ be positive constants and consider the surface Σ that is the part of the cylinder defined by $\Sigma = \{(x,y,z) : x^2 + z^2 = b^2 \text{ and } -\ell \leq y \leq \ell\}$. Calculate the total mass flux of the fluid flow with constant density $\rho(\boldsymbol{r}) = \rho_0$ through the cylinder Σ for the following flow velocities, where k is a constant (with units of s^{-1}).

   (a) $\boldsymbol{v} = (0, 0, kz)$

   (b) $\boldsymbol{v} = (kx, ky, kz)$

4. If a scalar field $T : \mathbb{R}^3 \to \mathbb{R}$ represents a temperature distribution, the *heat flux density* (the flow of energy per unit of area per unit of time) corresponding to this temperature distribution is the vector field $\boldsymbol{F} = -k\nabla T$, where k is the thermal conductivity of the material (in units of watts per meter-kelvin in SI units). The *total heat flux* through a surface is the integral of the heat flux density across that surface.

   Suppose $T(x,y,z) = x^2 + y^2 + z^2$ represents the temperature in a region of space around the origin of the coordinate system. Compute the total heat flux across the unit sphere.

5. Let $\boldsymbol{F}(x,y,z) = (xz, yz, x^2 + y^2)$. Find the outward flux of $\boldsymbol{F}$ across the boundary surface of the solid given by $x^2 + y^2 \leq z \leq 1$. Hint: there are two separate parts of the surface.

6. Find the divergence and curl of the following vector fields:

(a) $\boldsymbol{F}(x, y, z) = (x - 2z)\,\hat{\boldsymbol{\imath}} + (x + y + z)\,\hat{\boldsymbol{\jmath}} + (x - 2y)\,\hat{\boldsymbol{k}}$

(b) $\boldsymbol{F}(x, y, z) = e^x \sin y\,\hat{\boldsymbol{\imath}} + e^x \cos y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$

7. In this problem you will prove two important results.

   (a) Show that if $f : \Omega \to \mathbb{R}$ has continuous second-order partial derivatives on $\Omega \subseteq \mathbb{R}^3$, then

   $$\nabla \times (\nabla f) = \mathbf{0}.$$

   (b) Show that if $\boldsymbol{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ and P, Q, R have continuous second-order partial derivatives, then

   $$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0.$$

8. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $\boldsymbol{F}(x, y, z) = (xy, yz, zx)$. Let Γ be the curve defined as the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above. Use Stokes' Theorem to compute the circulation of the field $\boldsymbol{F}$ around the curve Γ. (Hint: on which surface does the curve Γ lie?)

9. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $\boldsymbol{F} = (y - z, -x - z, x + y)$ for all $(x, y, z) \in \mathbb{R}^3$

   (a) Use Stokes' Theorem to evaluate

   $$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA,$$

   where the surface Σ is the portion of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$ and $\hat{\boldsymbol{n}}$ is the upward-pointing unit normal.

   (b) Let Σ be the disk of radius 3 on the xy-plane centered at the origin, with unit normal vector $\hat{\boldsymbol{n}}$ pointing in the positive z-direction. Calculate

   $$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA$$

   from the definition of the surface integral. (Note that the surface can be given explicity as $\Sigma = \{(x, y, z) : x^2 + y^2 \leq 9 \text{ and } z = 0\}$).

   (c) What is the connection between (a) and (b)?