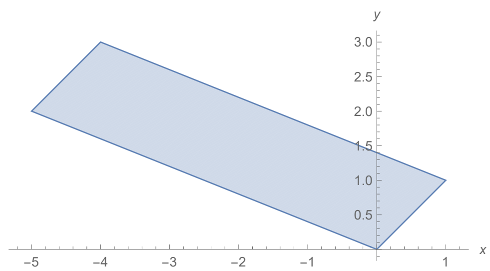
ECE 206 Fall 2019
Practice Problems Week 5
Solutions

1. Use a coordinate transformation to evaluate the following integrals. Make sure to sketch the region of integration in each case.

   (a) The integral $\iint_D (x+y)\,dx\,dy$, where D is the trapezoidal region with vertices given by $(0,0)$, $(1,1)$, $(-4,3)$, and $(-5,2)$.
   (Hint: use the coordinate transformations $x(u,v) = -5u + v$ and $y(u,v) = 2u + v$.)
   Solution. The region D can be visualized as in the following figure:



While it would be possible to compute the integral straightforwardly, a simpler way to compute it would be to make the suggested change of coordinates.
Note that every point in D can be written as a linear combination

$$\begin{pmatrix} x \\ y \end{pmatrix} = u \begin{pmatrix} -5 \\ 2 \end{pmatrix} + v \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for some values $u \in [0,1]$ and $v \in [0,1]$. (Indeed, the vectors $(-5,2)$ and $(1,1)$ determine the sides of the parallelogram D.) The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -5 & 1 \\ 2 & 1 \end{vmatrix} = (-5)(1) - (2)(1) = -7.$$

The desired integral can therefore be written as

$$\begin{aligned}
\iint_D (x+y)\,dx\,dy &= \int_0^1 \int_0^1 \big((-5u+v) + (2u+v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv \\
&= 7 \int_0^1 \int_0^1 (2v - 3u)\,du\,dv \\
&= 7 \int_0^1 \left[2uv - \frac{3}{2}u^2 \right]_{u=0}^1 dv \\
&= 7 \int_0^1 \left(2v - \frac{3}{2} \right) dv \\
&= 7 \left[v^2 - \frac{3}{2}v \right]_{v=0}^1 = 7\left(1 - \frac{3}{2}\right) = \frac{7}{2}.
\end{aligned}$$

(b) Find the volume of the solid under the paraboloid defined by $z = 2 - x^2 - y^2$, above the xy-plane, and inside the cylinder defined by $x^2 + y^2 = 1$.

Solution. The solid lies entirely above the circle $x^2 + y^2 = 1$ in the xy-plane, which we call D. The solid is the region above D between the plane $z = 0$ and the paraboloid $z = 2 - x^2 - y^2$. The volume is therefore given by

$$V = \iint_D \left(2 - x^2 - y^2\right)\, dx\, dy$$

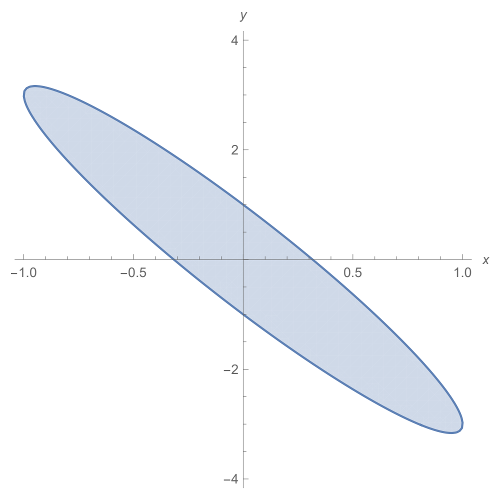
Use polar coordinates, the integral becomes

$$V = \int_0^{2\pi} \int_0^1 (2 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \left[r^2 - \frac{r^4}{4} \right]\Big|_{r=0}^1 d\theta = \left(1 - \frac{1}{4}\right)(2\pi) = \frac{3\pi}{2}.$$

(c) The integral $\iint_D x^2\, dx\, dy$, where D is the region inside the ellipse $10x^2 + 6xy + y^2 = 2$.

Use the coordinate transformations $x(u, v) = \sqrt{2}u$ and $y(u, v) = \sqrt{2}(v - 3u)$ to and verify that this transforms the unit circle in the uv-plane to the region D in the xy-plane. Then use polar coordinates compute the resulting integral.)

Solution. The region D may be visualized as shown in the following diagram.



We now need to find the $\mathbf{\Phi}^{-1}(D)$ corresponding to the parametrization defined in the problem as $\mathbf{\Phi}(u, v) = (\sqrt{2}u, \sqrt{2}(v - 3u))$.

Note: We must first discuss where this parametrization comes from. Plugging in the parameterization $x(u, v) = \sqrt{2}u$ and $y(u, v) = \sqrt{2}(v - 3u)$ into the equation $10x^2 + 6xy + y^2 = 2$ defining the ellipse, we find

$$\begin{aligned} 10x(u,v)^2 + 6x(u,v)y(u,v) + y(u,v)^2 &= 10(\sqrt{2}u)^2 + 6(\sqrt{2}u)(\sqrt{2}(v - 3u)) + (\sqrt{2}(v - 3u))^2 \\ &= 20u^2 + 12(uv - 3u^2) + 2(v^2 - 6uv + 9u^2) \\ &= 2(u^2 + v^2) \end{aligned}$$

ore equivalently $u^2 + v^2 = 1$ which is the circle in the uv-plane. Hence $\mathbf{\Phi}$ maps the circle in the uv-plane to te ellipse D in the xy-plane, as desired.

The Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{vmatrix} = 2.$$

We can therefore write our original integral in the uv-plane as follows

$$\iint_D x^2\, dA = \iint_{\mathbf{\Phi}^{-1}(D)} \left(\sqrt{2}u\right)^2 (2)\, du\, dv = 4 \iint_{\mathbf{\Phi}^{-1}(D)} u^2\, du\, dv,$$

where $\mathbf{\Phi}^{-1}(D)$ is the unit circle in the uv-plane.

The final step is to transform to polar coordinates, by letting $u = r\cos\theta$ and $v = r\sin\theta$. This yields

$$\begin{aligned} 4 \iint_{\mathbf{\Phi}^{-1}(D)} u^2\, du\, dv &= 4 \int_0^{2\pi} \int_0^1 (r\cos\theta)^2\, (r)\, dr\, d\theta \\ &= 4 \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta\, dr\, d\theta \\ &= 4 \int_0^{2\pi} \left(\frac{1+\cos(2\theta)}{2}\right) \frac{r^4}{4}\bigg|_{r=0}^{1} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1+\cos(2\theta))\, d\theta \\ &= \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\bigg|_{\theta=0}^{2\pi} = \pi. \end{aligned}$$

2. Let Γ be the closed curve consisting of the semi-circle $x^2 + y^2 = 9$ (with $y \geq 0$) and the x-axis from -3 to 3, oriented in the clockwise direction, and let the vector field $\boldsymbol{F}$ be defined by $\boldsymbol{F} = (x^2 y, -xy^2)$. Compute $\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$. (Hint: Use Green's theorem, then an appropriate change of coordinates.)

Solution. Notice that the curve is oriented clockwise, so we must put a negative sign when using Green's theorem to find that

$$\begin{aligned} \oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} &= -\oint_{-\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = -\iint_D (-y^2 - x^2)\, dx\, dy = \iint_D (x^2 + y^2)\, dx\, dy. \end{aligned}$$

Switching to polar coordinates, the value of the double integral can be evaluated as

$$\iint_D (x^2 + y^2)\, dx\, dy = \int_0^\pi \int_0^3 (r^2)\, r\, dr\, d\theta = \pi \frac{r^4}{4}\bigg|_{r=0}^{3} = \frac{81\pi}{4}.$$

3. Suppose a vector field $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ represents the velocity of the flow of some fluid moving through space with mass density given by the scalar field $\rho : \mathbb{R}^3 \to \mathbb{R}$ (in units of kg per meters squared). The total *mass flux* of the fluid flow is the vector field $\boldsymbol{F}(\boldsymbol{r}) = \rho(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r})$, and the *total mass flux* through a surface Σ is the surface integral $\iint_\Sigma \rho\boldsymbol{v} \cdot d\boldsymbol{A}$.

   Let $b, \ell > 0$ be positive constants and consider the surface Σ that is the part of the cylinder defined by $\Sigma = \{(x, y, z) : x^2 + z^2 = b^2 \text{ and } -\ell \leq y \leq \ell\}$. Calculate the total mass flux of the fluid flow with constant density $\rho(\boldsymbol{r}) = \rho_0$ through the cylinder Σ for the following flow velocities, where k is a constant (with units of s^{-1}).

   (a) $\boldsymbol{v} = (0, 0, kz)$

Solution. The mass flux vector is $\boldsymbol{F} = (0, 0, \rho_0 kz)$. The cylinder can be parameterized by $\boldsymbol{\Phi}(u, v) = (b\cos v, u, b\sin v)$, where $-\ell \le u \le \ell$ and $0 \le v \le 2\pi$. We want to compute

$$\iint_\Sigma \boldsymbol{F} \cdot d\boldsymbol{A} = \iint_{D_{uv}} \boldsymbol{F}(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right) du\, dv$$

where

$$\begin{aligned}\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ -b\sin v & 0 & b\cos v \end{vmatrix} \\ &= (b\cos v)\hat{\boldsymbol{\imath}} + (b\sin v)\hat{\boldsymbol{k}}.\end{aligned}$$

Notice that the normal is pointing outward from the surface of the cylinder. Note that $\boldsymbol{F}(\boldsymbol{\Phi}(u,v)) = (0, 0, \rho_0 kb\sin v)$ and thus

$$\boldsymbol{F}(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial v} \times \frac{\partial \boldsymbol{\Phi}}{\partial u}\right) = \rho_0 kb^2 \sin^2 v.$$

The surface integral can therefore be computed as

$$\begin{aligned}\iint_\Sigma \boldsymbol{F} \cdot d\boldsymbol{A} &= \rho_0 kb^2 \int_{-\ell}^{\ell} \int_0^{2\pi} \sin^2 v\, dv\, du \\ &= \rho_0 kb^2 \left(\int_{-\ell}^{\ell} du\right)\left(\int_0^{2\pi} \frac{1-\cos(2v)}{2}\, dv\right) \\ &= 2\pi\rho_0 kb^2 \ell.\end{aligned}$$

Note that our answer has dimensions of mass per unit time, as expected.

(b) $\boldsymbol{v} = (kx, ky, kz)$

Solution. Now, $\boldsymbol{F}(\boldsymbol{\Phi}(u,v)) = \rho_0 k(b\cos v, u, b\sin v)$. Then

$$\boldsymbol{F}(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right) = \rho_0 kb^2(\cos^2 v + \sin^2 v) = \rho_0 kb^2$$

The flux can therefore be computed as

$$\iint_{D_{uv}} \boldsymbol{F}(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right) du\, dv = \rho_0 kb^2 \int_{-\ell}^{\ell} \int_0^{2\pi} dv\, du = 4\pi\rho_0 kb^2 \ell.$$

4. If a scalar field $T : \mathbb{R}^3 \to \mathbb{R}$ represents a temperature distribution, the *heat flux density* (the flow of energy per unit of area per unit of time) corresponding to this temperature distribution is the vector field $\boldsymbol{F} = -k\nabla T$, where k is the thermal conductivity of the material (in units of watts per meter-kelvin in SI units). The *total heat flux* through a surface is the integral of the heat flux density across that surface.

   Suppose $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperature in a region of space around the origin of the coordinate system. Compute the total heat flux across the unit sphere.

Solution. The heat flux density vector is $\boldsymbol{J} = -k\nabla T = -2k(x, y, z)$. We can parameterize the surface of the sphere Σ by $\boldsymbol{\Phi}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$ for $u \in [0, \pi]$ and $v \in [0, 2\pi]$. The heat flux is

$$J = \iint_\Sigma \boldsymbol{J} \cdot d\boldsymbol{A} = \iint_\Sigma \boldsymbol{J} \cdot \hat{\boldsymbol{n}}\, dA = \iint_{D_{uv}} \boldsymbol{J}(\boldsymbol{\Phi}(u, v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right) du\, dv$$

where the partial derivatives of $\boldsymbol{\Phi}$ are computed as

$$\frac{\partial \boldsymbol{\Phi}}{\partial u} = (\cos u \cos v, \cos u \sin v, -\sin u) \qquad \text{and} \qquad \frac{\partial \boldsymbol{\Phi}}{\partial v} = (-\sin u \sin v, \sin u \cos v, 0)$$

such that the normal vector is given by

$$\begin{aligned} \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} \\ &= (\sin^2 u \cos v,\ \sin^2 u \sin v,\ \sin u \cos u). \end{aligned}$$

We have that

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{\Phi}(u, v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right) &= -2k(\sin u \cos v, \sin u \sin v, \cos u) \cdot (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u) \\ &= -2k \sin u\, (\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u) \\ &= -2k \sin u\, (\sin^2 u\, (\cos^2 v + \sin^2 v) + \cos^2 u) \\ &= -2k \sin u\, (\sin^2 u + \cos^2 u) \\ &= -2k \sin u. \end{aligned}$$

The heat flux across the unit sphere S is therefore computed as

$$J = \iint_\Sigma \boldsymbol{J} \cdot d\boldsymbol{A} = -2k \int_0^\pi \int_0^{2\pi} \sin u\, dv\, du = -2k(2\pi)(-\cos u)\Big|_{u=0}^{\pi} = -8k\pi.$$

(Note on the sign: since the temperature increases radially outwards, and heat flows from hot to cold, we expect that heat will be transferred across the surface in the direction of $-\hat{\boldsymbol{n}}$, and thus that the heat flux will be negative.)

5. Let $\boldsymbol{F}(x, y, z) = (xz, yz, x^2 + y^2)$. Find the outward flux of $\boldsymbol{F}$ across the boundary surface of the solid given by $x^2 + y^2 \leq z \leq 1$. Hint: there are two separate parts of the surface.

Solution. The surface consists of the paraboloid $z = x^2 + y^2$ (call it Σ_1) as well as the top of the solid region, which is the disk $x^2 + y^2 = 1$ in the $z = 1$ plane (Σ_2).
Let's start with the disk Σ_2. This can be parameterized by $\boldsymbol{\Phi}_2(u, v) = (u \cos v, u \sin v, 1)$ for $u \in [0, 1]$ and $v \in [0, 2\pi]$. The partial derivatives of $\boldsymbol{\Phi}$ are given by

$$\frac{\partial \boldsymbol{\Phi}_2}{\partial u} = (\cos v, \sin v, 0) \qquad \text{and} \qquad \frac{\partial \boldsymbol{\Phi}_2}{\partial v} = (-u \sin v, u \cos v, 0)$$

such that the corresponding normal vector to the surface at any point on the surface is equal to

$$\frac{\partial \boldsymbol{\Phi}_2}{\partial u} \times \frac{\partial \boldsymbol{\Phi}_2}{\partial v} = (0, 0, u).$$

The field at each point on the parameterized surface is given by $\boldsymbol{F}(\boldsymbol{\Phi}_2(u,v)) = (u\cos v, u\sin v, u^2)$. The flux through Σ_2 is therefore computed as

$$\iint_{\Sigma_2} \boldsymbol{F}\cdot d\boldsymbol{A} = \int_0^{2\pi}\int_0^1 (u\cos v, u\sin v, u^2)\cdot(0,0,u)\,du\,dv = \int_0^{2\pi} dv \int_0^1 u^3\,du = \frac{\pi}{2}.$$

The paraboloid Σ_1 can be parameterized by $\boldsymbol{\Phi}_1(u,v) = (u\cos v, u\sin v, u^2)$ for $u\in[0,1]$ and $v\in[0,2\pi]$. Then

$$\frac{\partial \boldsymbol{\Phi}_1}{\partial u} = (\cos v, \sin v, 2u), \qquad \frac{\partial \boldsymbol{\Phi}_1}{\partial v} = (-u\sin v, u\cos v, 0)$$

and thus

$$\frac{\partial \boldsymbol{\Phi}_1}{\partial u}\times\frac{\partial \boldsymbol{\Phi}_1}{\partial v} = (-2u^2\cos v, -2u^2\sin v, u).$$

Note here that the normal points upward (the z-coordinate is non-negative), which is incorrect since that is the inward normal. So we simply switch u and v, or take a negative sign from the cross-product, to find the correct normal vector. The field at each point on Σ_1 is given by $\boldsymbol{F}(\boldsymbol{\Phi}_1(u,v)) = (u^3\cos v, u^3\sin v, u^2)$. The flux through Σ_1 is therefore computed as

$$\begin{aligned}\iint_{\Sigma_1} \boldsymbol{F}\cdot d\boldsymbol{A} &= \int_0^{2\pi}\int_0^1 (u^3\cos v, u^3\sin v, u^2)\cdot(2u^2\cos v, 2u^2\sin v, -u)\,du\,dv \\ &= \int_0^{2\pi} dv\int_0^1 2u^5 - u^3\,du = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6}\end{aligned}$$

Thus, the total flux through the boundary is

$$\iint_{\Sigma_1}\boldsymbol{F}\cdot d\boldsymbol{A} + \iint_{\Sigma_2}\boldsymbol{F}\cdot d\boldsymbol{A} = \frac{\pi}{2}+\frac{\pi}{6} = \frac{2\pi}{3}$$

6. Find the divergence and curl of the following vector fields:

(a) $\boldsymbol{F}(x,y,z) = (x-2z)\,\hat{\boldsymbol{\imath}} + (x+y+z)\,\hat{\boldsymbol{\jmath}} + (x-2y)\,\hat{\boldsymbol{k}}$

Solution. The divergence is computed as

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(x-2z) + \frac{\partial}{\partial y}(x+y+z) + \frac{\partial}{\partial z}(x-2y) = 1+1+0 = 2$$

while the curl is given by

$$\begin{aligned}\nabla\times\boldsymbol{F} &= \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-2z & x+y+z & x-2y\end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(x-2y) - \frac{\partial}{\partial z}(x+y+z)\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial}{\partial z}(x-2z) - \frac{\partial}{\partial x}(x-2y)\right)\hat{\boldsymbol{\jmath}} \\ &\qquad + \left(\frac{\partial}{\partial x}(x+y+z) - \frac{\partial}{\partial y}(x-2z)\right)\hat{\boldsymbol{k}} \\ &= (-2-1)\hat{\boldsymbol{\imath}} + (-2-1)\hat{\boldsymbol{\jmath}} + (1-0)\hat{\boldsymbol{k}} \\ &= -3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}.\end{aligned}$$

(b) $\boldsymbol{F}(x,y,z) = e^x \sin y\,\hat{\boldsymbol{\imath}} + e^x \cos y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$

Solution. The divergence is computed as

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(z) = e^x \sin y - e^x \sin y + 1 = 1$$

while the curl is given by

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(e^x \cos y)\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial}{\partial z}(e^x \sin y) - \frac{\partial}{\partial x}(z)\right)\hat{\boldsymbol{\jmath}} \\
&\qquad + \left(\frac{\partial}{\partial x}(e^x \cos y) - \frac{\partial}{\partial y}(e^x \sin y)\right)\hat{\boldsymbol{k}} \\
&= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + (e^x \cos y - e^x \cos y)\hat{\boldsymbol{k}} \\
&= \boldsymbol{0}.
\end{aligned}$$

7. In this problem you will prove two important results.

(a) Show that if $f : \Omega \to \mathbb{R}$ has continuous second-order partial derivatives on $\Omega \subseteq \mathbb{R}^3$, then

$$\nabla \times (\nabla f) = \boldsymbol{0}.$$

Solution. Note that $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. The curl of the gradient field is therefore computed as

$$\begin{aligned}
\nabla \times (\nabla f) &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= (f_{yz} - f_{zy})\,\hat{\boldsymbol{\imath}} + (f_{zx} - f_{xz})\,\hat{\boldsymbol{\jmath}} + (f_{xy} - f_{yx})\,\hat{\boldsymbol{k}} \\
&= \boldsymbol{0}
\end{aligned}$$

by the equality of mixed partial derivatives, which is guaranteed by the continuity of second order partials of f. This shows that any conservative vector field is irrotational. That is, if $\boldsymbol{F} = \nabla\Psi$ for some scalar field Ψ, it holds that $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

(b) Show that if $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ and P, Q, R have continuous second-order partial derivatives, then

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0.$$

Solution. We first compute the curl of $\boldsymbol{F}$ as

$$\begin{aligned}\nabla \times \boldsymbol{F} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\boldsymbol{\imath}}.\end{aligned}$$

The divergence of the curl is computed as

$$\begin{aligned}\nabla \cdot (\nabla \times \boldsymbol{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\ &= R_{xy} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} \\ &= 0,\end{aligned}$$

where we make use of the equality of mixed partial derivatives (e.g. $R_{xy} = R_{yx}$). This shows that any vector field which is the curl of some other vector field is solenoidal. That is, if $\boldsymbol{A} = \nabla \times \boldsymbol{F}$, then $\nabla \cdot \boldsymbol{A} = 0$. In this case we call $\boldsymbol{F}$ a *vector potential* for $\boldsymbol{A}$. It turns out that this implication goes the other way as well, as the following theorem states.

Theorem. If $\nabla \cdot \boldsymbol{A} = 0$, then there exists a vector field $\boldsymbol{F}$ such that $\boldsymbol{A} = \nabla \times \boldsymbol{F}$.

8. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $\boldsymbol{F}(x, y, z) = (xy, yz, zx)$. Let Γ be the curve defined as the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above. Use Stokes' Theorem to compute the circulation of the field $\boldsymbol{F}$ around the curve Γ. (Hint: on which surface does the curve Γ lie?)

Solution. The curve lies on the plane passing through those same points, which turns out to be the plane $x + y + z = 1$. We can parameterize the plane using the parameterization $\boldsymbol{\Phi}(u, v) = (u, v, 1 - u - v)$. Let Σ denote the surface that is the region contained inside the triangular curve Γ on this plane. To find the bounds on the parameter varialbes u and v that define the surface Σ, we see that it corresponds to the region in the xy-plane that lies below Σ, which is a triangle. Hence the surface Σ can be parameterized by $\boldsymbol{\Phi}$ with the domain defined by

$$0 \le u \le 1, \quad 0 \le v \le 1 - u.$$

By Stokes' Theorem, it holds that

$$\begin{aligned}\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} &= \iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA \\ &= \iint_{D_{uv}} (\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(u, v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right) du\, dv,\end{aligned}$$

where D_{uv} is the region in the parameter space that parameterizes Σ. Note that the curl of $\boldsymbol{F}$ is computed as$\nabla \times \boldsymbol{F} = (-y, -z, -x)$, and the term $(\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(u, v))$ means we evaluate the curl expression at the parameterized point on the surface, which yields

$$(\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(u, v)) = (-v, u + v - 1, -u).$$

To compute the normal vector, we have that

$$\frac{\partial \boldsymbol{\Phi}}{\partial u} = (1, 0, -1), \quad \frac{\partial \boldsymbol{\Phi}}{\partial v} = (0, 1, -1) \quad \text{and thus} \quad \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} = (1, 1, 1).$$

The integrand is therefore given by

$$(\nabla \times \boldsymbol{F})(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right) = -v + (u+v-1) - u = -1$$

The circulation is therefore computed as

$$\int_0^1 \int_0^{1-u} (-1)\, dv\, du = -\int_0^1 (1-u)\, du = -\left(u - \frac{u^2}{2} \right)\Big|_{u=0}^1 = -\frac{1}{2}.$$

Solution. (**Alternate solution**) From the equation of the plane, we know a (unnormalized) normal vector is $\boldsymbol{n} = (1,1,1)$, so that the unit normal is given by

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}(1,1,1).$$

We may now evaluate $(\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}$ directly as

$$(\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}} = (-y,-z,-x) \cdot \frac{1}{\sqrt{3}}(1,1,1) = -\frac{1}{\sqrt{3}}(x+y+z) = -\frac{1}{\sqrt{3}},$$

since we are on the plane $x+y+z=1$. We therefore have that

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = -\frac{1}{\sqrt{3}} \iint_\Sigma dA = -\frac{1}{\sqrt{3}} \operatorname{Area}(\Sigma)$$

where $\operatorname{Area}(\Sigma)$ denotes the surface area of Σ. The surface is the triangle, with base the line segment from $(1,0,0)$ to $(0,1,0)$, which has length $\sqrt{2}$. The height can be measured from the midpoint of the base $(\frac{1}{2}, \frac{1}{2}, 0)$ to the point $(0,0,1)$, which turns out to be $\sqrt{\frac{3}{2}}$. Thus, our final answer is

$$-\frac{1}{\sqrt{3}} S(\Sigma) = -\frac{1}{\sqrt{3}} \left(\frac{1}{2}\sqrt{2}\sqrt{\frac{3}{2}} \right) = -\frac{1}{2}$$

as above.

9. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $\boldsymbol{F} = (y-z, -x-z, x+y)$ for all $(x,y,z) \in \mathbb{R}^3$

   (a) Use Stokes' Theorem to evaluate

   $$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA,$$

   where the surface Σ is the portion of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$ and $\hat{\boldsymbol{n}}$ is the upward-pointing unit normal.

   Solution. By Stokes' Theorem, we have that

   $$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = \oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r}$$

   where $\partial\Sigma$ is the curve $x^2+y^2=9$ on the plane $z=0$, which can be parameterized by $\boldsymbol{\gamma}(t) = (3\cos t, 3\sin t, 0)$, $0 \leq t \leq 2\pi$. It follows that $\boldsymbol{\gamma}'(t) = (-3\sin t, 3\cos t, 0)$. The vector field evaluated at each point on the curve is given by $\boldsymbol{F}(\boldsymbol{\gamma}(t)) = (3\sin t, -3\cos t, 3\cos t + 3\sin t)$.

The desired integral is therefore computed as

$$\begin{aligned}\oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{2\pi} (3\sin t, -3\cos t, 3\cos t + 3\sin t) \cdot (-3\sin t, 3\cos t, 0)\, dt \\ &= \int_0^{2\pi} -9(\sin^2 t + \cos^2 t)\, dt = -18\pi.\end{aligned}$$

(b) Let Σ be the disk of radius 3 on the xy-plane centered at the origin, with unit normal vector $\hat{\boldsymbol{n}}$ pointing in the positive z-direction. Calculate

$$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA$$

from the definition of the surface integral. (Note that the surface can be given explicity as $\Sigma = \{(x, y, z) : x^2 + y^2 \leq 9 \text{ and } z = 0\}$).

Solution. The curl of $\boldsymbol{F}$ is computed as $\nabla \times \boldsymbol{F} = (2, -2, -2)$. There are now two ways to proceed.

i. We may notice that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, and thus $(\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}} = (2, -2, -2) \cdot (0, 0, 1) = -2$. The desired integral may therefore be computed as

$$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = -2 \iint_\Sigma dA = -2\,\text{Area}(S) = -18\pi,$$

where the surface area of the disk of radius 3 is $\text{Area}(S) = 9\pi$.

ii. On the other hand, we may parameterize S using the parameterization $\boldsymbol{\Phi}(u, v) = (u\cos v, u\sin v, 0)$ over the domain $u \in [0, 3]$ and $v \in [0, 2\pi]$. Computing the partial derivatives and taking the cross product, we find

$$\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} = (0, 0, u).$$

The desired integral can therefore be computed as

$$\begin{aligned}\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA &= \int_0^{2\pi} \int_0^3 (2, -2, -2) \cdot (0, 0, u)\, du\, dv \\ &= -2 \int_0^{2\pi} \int_0^3 u\, du\, dv = -18\pi.\end{aligned}$$

(c) What is the connection between (a) and (b)?

Solution. They are equal, which can be explained as follows.

Suppose we have two oriented surfaces Σ_1 and Σ_2 such that $\partial\Sigma_1 = \partial\Sigma_2 = \Gamma$, where Γ is a simple closed curve. Then

$$\iint_{\Sigma_1} (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}_1\, dA = \iint_{\Sigma_2} (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}_2\, dA.$$

In fact, given a simple closed curve Γ one can imagine infinitely many surfaces Σ with

$\partial\Sigma = \Gamma$. In this way, the value of the surface integral is independent of which surface Σ we choose. Thus, we get the analogy of a surface integral being *independent of surface*, much as a line integral can be independent of path.

The key point here is that the vector field must be of the form $\nabla \times \boldsymbol{F}$. In other words, it must be the curl of some other vector field. So, given a vector field $\boldsymbol{G}$, how do we tell if $\boldsymbol{G} = \nabla \times \boldsymbol{F}$, for some other vector field $\boldsymbol{F}$?

Well, we've seen previously that $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$ for any $\boldsymbol{F}$, so the requirement is $\nabla \cdot \boldsymbol{G} = 0$. If $\boldsymbol{F}$ exists such that $\boldsymbol{G} = \nabla \times \boldsymbol{F}$, we say that $\boldsymbol{F}$ is a *vector potential* for $\boldsymbol{G}$.