ECE 206 Fall 2019
Practice Problems Week 6

1. The *mass* of a volume $\Omega \in \mathbb{R}^3$ is defined as $M(\Omega) = \iiint_\Omega \sigma(x,y,z)\,dV$ where $\sigma : \Omega \to \mathbb{R}$ is the mass density function (i.e., mass per unit volume). The *centre of gravity* of Ω is the point $\boldsymbol{p}_c = (x_c, y_c, z_c)$ in $\mathbb{R}^3$ defined by the integrals

$$x_c = \frac{1}{M(\Omega)} \iiint_\Omega x\,\sigma(x,y,z)\,dV, \quad y_c = \frac{1}{M(\Omega)} \iiint_\Omega y\,\sigma(x,y,z)\,dV, \quad z_c = \frac{1}{M(\Omega)} \iiint_\Omega z\,\sigma(x,y,z)\,dV.$$

For each of the regions described below, make a sketch of the region and compute the mass and centre of gravity.

   (a) Let Ω be the tetrahedral region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 2z = 4$. Find the mass and centre of gravity of Ω, assuming it has constant density $\sigma(x,y,z) = 1$.

   (b) Let Ω be the ice-cream cone-shaped region inside the unit hemisphere $z = \sqrt{1 - x^2 - y^2}$ and inside the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$. Find the mass and centre of gravity of Ω, assuming it has constant density $\sigma(x,y,z) = k$.

   (c) Let Ω be the ice-cream cone-shaped region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and inside the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$. Suppose the density at any point is given by $\sigma(x,y,z) = 2 - z$. Find the mass and centre of gravity.

   (d) Let Ω be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the total mass and centre of gravity of Ω.

2. Find the volume of the the region inside the unit hemisphere $z = \sqrt{1 - x^2 - y^2}$ but **outside** the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$.

3. Determine the curl and divergence of the vector field defined for all $\boldsymbol{r} \neq \boldsymbol{0}$ as $\boldsymbol{F}(\boldsymbol{r}) = \frac{\boldsymbol{r}}{r^p}$, where $\boldsymbol{r} = (x,y,z)$ and $r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, and $p > 0$ is a constant.

4. A repeat from the last set of practice problems: Let $b, k, \ell > 0$ be positive constants and consider the region defined by $\Omega = \{(x,y,z) : x^2 + z^2 \leq b^2 \text{ and } -\ell \leq y \leq \ell\}$ that is inside the cylinder $\{(x,y,z) : x^2 + z^2 = b^2 \text{ and } -\ell \leq y \leq \ell\}$.

   (a) Let a fluid flow have velocity $\boldsymbol{v} = (0, 0, kz)$ and constant density ρ_0. Compute the integral $\iiint_\Omega \nabla \cdot \boldsymbol{F}\,dV$, where $\boldsymbol{F} = \rho_0 \boldsymbol{v}$ is the mass flux vector.

   (b) Compute $\iiint_\Omega \nabla \cdot \boldsymbol{F}\,dV$, this time with $\boldsymbol{v} = (kx, ky, kz)$.

   (c) Did you get the same answers as those in the last assignment for the flux? Explain why or why not, and verify Gauss' Theorem by completing any necessary calculations.

5. For the following vector fields $\boldsymbol{F}$, determine whether or not they are solenoidal and find a corresponding vector potential field $\boldsymbol{G}$ if it exists.

(a) $\boldsymbol{F}(x, y, z) = (yz, xz, xy)$

(b) $\boldsymbol{F}(x, y, z) = (xz^2 - 1,\ -yz^2,\ 1 - x^2)$

(c) $\boldsymbol{F}(x, y, z) = (x, y, z)$

6. Find the flux of the vector field $\boldsymbol{F}(x, y, z) = (xz, -yz, 1 + y^2)$ across the surface Σ that is defined as $\Sigma = \{(x, y, z) \,|\, z = \cos^{-1}(x^2 + y^2) \text{ and } x^2 + y^2 \leq 1\}$ with outward pointing normal.

   Hint: Do not attempt to do this directly. Instead, notice that $\nabla \cdot \boldsymbol{F} = 0$ which implies there is a vector potential, call it $\boldsymbol{G}$, such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$. Also, note that vector potentials are not unique, and therefore you can choose any $\boldsymbol{G}$ that satisfies the above.

7. Derive the following vector calculus identity. For a vector field $\boldsymbol{F}$, it holds that

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$$

   where $\nabla^2 \boldsymbol{F}$ is the *vector Laplacian* of $\boldsymbol{F}$ defined as $\nabla^2 \boldsymbol{F} = \big(\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3\big)$.

8. Gauss' Law: The electric field $\boldsymbol{E} : \{\boldsymbol{r} \in \mathbb{R}^3 : \boldsymbol{r} \neq \boldsymbol{0}\} \to \mathbb{R}$ due to a point charge Q at the origin is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{kQ}{r^3}\boldsymbol{r}$$

   where $\boldsymbol{r} = (x, y, z)$, $r = \|\boldsymbol{r}\|$, and $k = \dfrac{1}{4\pi\epsilon_0}$ is constant. Show that

$$\oiint_{\Sigma} \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA = \begin{cases} 0 & \text{if } \Sigma \text{ does not enclose the charge} \\ \dfrac{Q}{\epsilon_0} & \text{if } \Sigma \text{ encloses the charge} \end{cases}$$

   where Σ is an arbitrary smooth, closed surface. For the second case, you may assume that Σ is a sphere centered at the origin.

9. Consider the cube $\{(x, y, z) \,|\, -1 \leq x, y, z \leq 1\}$. Find the charge enclosed by the cube if the electric field is:

   (a) $\boldsymbol{E}(x, y, z) = (x, y, z)$

   (b) $\boldsymbol{E}(x, y, z) = (x^2, y^2, z^2)$

10. For each law below, write the law mathematically in terms of integrals. Then use theorems from vector calculus to derive a partial differential equation that holds at every point in space, each leading to Maxwell's equations in differential form.

    (a) Faraday's Law: *The circulation of an electric field* $\boldsymbol{E}$ *around the perimeter of a surface is equal to the negative time rate of change of the flux of the magnetic field* $\boldsymbol{B}$ *through the surface.*

    (b) No magnetic monopoles: *The flux of the magnetic field through any closed surface is zero.*

    (c) Ampere's Law: *The circulation of the magnetic field around the perimeter of a surface is equal to the time rate of change of the flux of the electric field through the surface + the flux of the electric current density through the surface* (where there are constants of proportionality).

Vector calculus is useful for all sorts of engineering and physics applications, not just electromagnetism. In particular, heat flow and fluid flow are modeled well by vector calculus. We will not go into these examples in any more detail in the course, but these problems are here to show you some of the further physical applications of vector calculus.

1. A volume Ω of a homogeneous and isotropic material is bounded by a smooth orientable surface $\partial\Omega$ and is being heated from outside. Since heat is energy and energy is conserved, it will be true that:

$$\text{(increase of heat in } \Omega) = \text{(flux of heat through } \partial\Omega) \tag{1}$$

Let the heat flux vector be denoted by $\boldsymbol{J}(\boldsymbol{r}, t)$, and the heat energy density be given by $\rho C_p T(\boldsymbol{r}, t)$, where ρ is the mass density, C_p is the heat capacity (both constant) and $T(\boldsymbol{r}, t)$ the temperature.

   (a) Using an appropriate theorem of integral vector calculus, translate (1) into a mathematical statement.

   (b) Fourier's heat says that $\boldsymbol{J} = -k\nabla T$ where the constant k represents the *thermal conductivity* of the material. Use this to derive a partial differential equation that $T(\boldsymbol{r}, t)$ must satisfy.

2. Consider a region Ω of a homogeneous solution with boundary $\partial\Omega$ (a smooth orientable surface). Suppose that a substance M is dissolved in the solution. Since mass is conserved, it will be true that:

$$\text{(rate of change of amount of } M \text{ in } \Omega) = -\,\text{(flux of } M \text{ through } \partial\Omega) \tag{2}$$

where the negative sign appears because a net flux outward corresponds to a decrease in the amount of M. Let the flux vector for M be denoted by $\boldsymbol{J}(\boldsymbol{r}, t)$, and let the concentration of M be $c(\boldsymbol{r}, t)$.

   (a) Using an appropriate theorem of integral vector calculus, translate (2) into a mathematical statement.

   (b) *Fick's Law* says that $\boldsymbol{J} = -D\nabla c$, where the constant of D is the *diffusion coefficient*. In words, this is "the flux of M is in the direction of $-\nabla c$, and its magnitude is proportional to $-\nabla c$." Use Fick's law to derive a partial differential equation that $c(\boldsymbol{r}, t)$ must satisfy.