ECE 206 Fall 2019
Practice Problems Week 6
Solutions

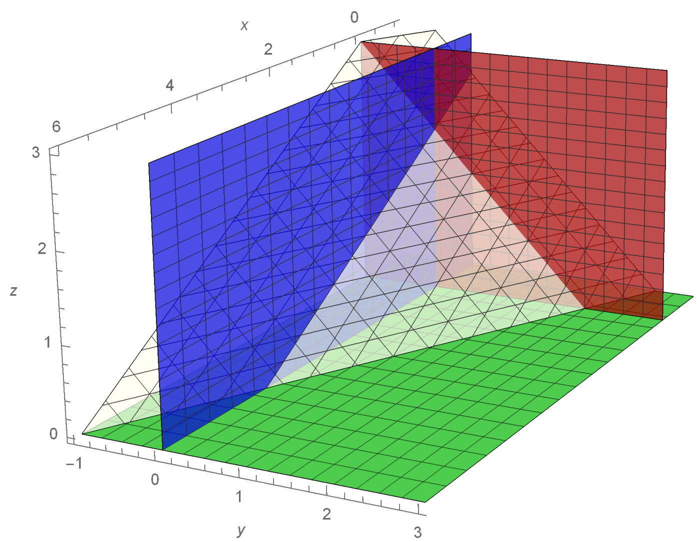
1. The *mass* of a volume $\Omega \in \mathbb{R}^3$ is defined as $M(\Omega) = \iiint_\Omega \sigma(x, y, z)\, dV$ where $\sigma : \Omega \to \mathbb{R}$ is the mass density function (i.e., mass per unit volume). The *centre of gravity* of Ω is the point $\boldsymbol{p}_c = (x_c, y_c, z_c)$ in $\mathbb{R}^3$ defined by the integrals

$$x_c = \frac{1}{M(\Omega)} \iiint_\Omega x\, \sigma(x, y, z)\, dV, \quad y_c = \frac{1}{M(\Omega)} \iiint_\Omega y\, \sigma(x, y, z)\, dV, \quad z_c = \frac{1}{M(\Omega)} \iiint_\Omega z\, \sigma(x, y, z)\, dV.$$

For each of the regions described below, make a sketch of the region and compute the mass and centre of gravity.

(a) Let Ω be the tetrahedral region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 2z = 4$. Find the mass and centre of gravity of Ω, assuming it has constant density $\sigma(x, y, z) = 1$.

Solution. This region can be visualized as in the following figure.



The red, blue, and green planes are the planes $x = 0$, $y = 0$, and $z = 0$ respectively. Note that there are many ways to integrate over this volume. A few different ways are as follows:

$$\int_0^2 \int_0^{z-2} \int_0^{4-2y-2z} (\cdot)\, dx\, dy\, dz, \qquad \int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{2-z-\frac{1}{2}x} (\cdot)\, dy\, dz\, dx,$$

$$\int_0^2 \int_0^{4-2y} \int_0^{2-y-\frac{1}{2}x} (\cdot)\, dz\, dx\, dy$$

The order of integration we use will simplify the integral depending on what we are integrating.

The mass (or, equivalently, the volume) of the solid is

$$\begin{aligned}
M(\Omega) &= \int_0^2 \left(\int_0^{2-z} \left(\int_0^{4-2y-2z} 1\, dx \right) dy \right) dz \\
&= \int_0^2 \left(\int_0^{2-z} (4-2y-2z)\, dy \right) dz \\
&= \int_0^2 \left((2-z)y - y^2 \right) \Big|_{y=0}^{2-z} dz \\
&= \int_0^2 \left((4-2z)(2-z) - (2-z)^2 \right) dz \\
&= \int_0^2 \left(4 - 4z + z^2 \right) dz \\
&= \left(4z - 2z^2 + \frac{z^3}{3} \right) \Big|_{z=0}^{2} = 8 - 8 + \frac{8}{3} = \frac{8}{3}.
\end{aligned}$$

Meanwhile, the coordinates for the centre of gravity may be computed as

$$\begin{aligned}
x_c &= \frac{1}{M(\Omega)} \iiint_\Omega x\, dV \\
&= \frac{3}{8} \int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{2-z-\frac{1}{2}x} x\, dy\, dz\, dx \\
&= \frac{3}{8} \int_0^4 x \left(\int_0^{2-\frac{1}{2}x} \left(2 - \frac{1}{2}x - z \right) dz \right) dx \\
&= \frac{3}{8} \int_0^4 x \left(\left[\left(2 - \frac{1}{2}x \right) z - \frac{z^2}{2} \right]_0^{2-\frac{1}{2}x} \right) dx \\
&= \frac{3}{8} \int_0^4 x \left(2 - x + \frac{x^2}{8} \right) dx \\
&= \frac{3}{8} \left(x^2 - \frac{x^3}{3} + \frac{x^4}{32} \right) \Big|_0^4 \\
&= \frac{3}{8} \left(16 - \frac{64}{3} + 8 \right) \\
&= 1
\end{aligned}$$

Similar integrations can be carried out to find that

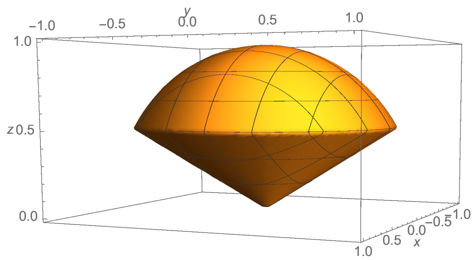
$$y_c = \frac{1}{M(\Omega)} \iiint_\Omega y\, dV = \frac{3}{8} \int_0^2 \int_0^{4-2y} \int_0^{2-y-\frac{1}{2}x} y\, dz\, dx\, dy = \frac{1}{2}$$

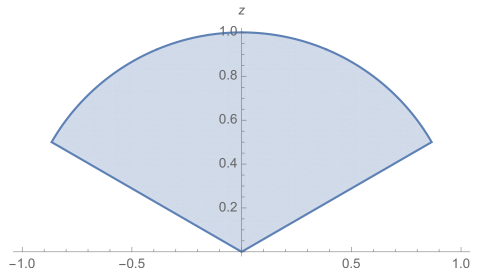
and

$$z_c = \frac{1}{M(\Omega)} \iiint_\Omega z\, dV = \frac{3}{8} \int_0^2 \int_0^{z-2} \int_0^{4-2y-2z} z\, dx\, dy\, dz = \frac{1}{2}.$$

(b) Let Ω be the ice-cream cone-shaped region inside the unit hemisphere $z = \sqrt{1-x^2-y^2}$ and inside the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$. Find the mass and centre of gravity of Ω, assuming it has constant density $\sigma(x, y, z) = k$.

Solution. This region can be visualized as in the following figure.





Using spherical coordinates, the mass of the solid can be computed as

$$
\begin{aligned}
M(\Omega) = \iiint_\Omega k\,dV &= k\int_0^{2\pi}\left(\int_0^{\frac{\pi}{3}}\left(\int_0^1 r^2\sin\varphi\,dr\right)d\varphi\right)d\theta \\
&= k\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r^2\,dr\right)\left(\int_0^{\frac{\pi}{3}}\sin\varphi\,d\varphi\right) \\
&= k(2\pi)\frac{r^3}{3}\bigg|_{r=0}^{1}(-\cos\varphi)\bigg|_{\varphi=0}^{\pi/3} \\
&= k\frac{\pi}{3}.
\end{aligned}
$$

By rotational symmetry, the center of gravity of the ice-cream cone will have $x_c = 0$ and $y_c = 0$. The z-coordinate for the centre of gravity is

$$
\begin{aligned}
z_c = \frac{1}{M(\Omega)}\iiint_\Omega kz\,dV &= \frac{3}{\pi}\int_0^{2\pi}\left(\int_0^{\frac{\pi}{3}}\left(\int_0^1 (r\cos\varphi)(r^2\sin\varphi)\,dr\right)d\varphi\right)d\theta \\
&= \frac{3}{\pi}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}}\cos\varphi\sin\varphi\,d\varphi\right)\left(\int_0^1 r^3\,dr\right) \\
&= \frac{3}{\pi}(2\pi)\left(\frac{\sin^2\varphi}{2}\right)\bigg|_{\varphi=0}^{\pi/3}\frac{r^4}{4}\bigg|_{r=0}^{1} \\
&= \frac{9}{16}.
\end{aligned}
$$

(c) Let Ω be the ice-cream cone-shaped region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and inside the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$. Suppose the density at any point is given by $\sigma(x, y, z) = 2 - z$. Find the mass and centre of gravity.

Solution. This region is almost the same as the region in part b, but the radius varies over $0 \le \rho \le 2$. Indeed, it is the region bound below by the same cone ($z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$), but bounded above by the hemisphere of radius 2 instead of radius 1. Using spherical coordinates, the mass of the solid can be computed as

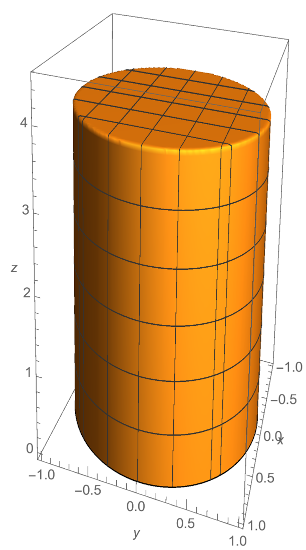
$$
\begin{aligned}
M(\Omega) &= \iiint_{\Omega} (2-z)\,dV = \int_0^{2\pi} \left(\int_0^{\frac{\pi}{3}} \left(\int_0^2 (2 - r\cos\varphi)(r^2\sin\varphi)\,dr \right) d\varphi \right) d\theta \\
&= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{3}} \left(\int_0^2 \left(2r^2\sin\varphi - r^3\cos\varphi\sin\varphi\right) dr \right) d\varphi \right) \\
&= (2\pi) \left(\int_0^{\frac{\pi}{3}} \left[\frac{2r^3}{3}\sin\varphi - \frac{r^4}{4}\cos\varphi\sin\varphi \right]_{r=0}^{2} d\varphi \right) \\
&= (2\pi) \int_0^{\frac{\pi}{3}} \left(\frac{16}{3}\sin\varphi - 4\cos\varphi\sin\varphi \right) d\varphi \\
&= (2\pi) \left[-\frac{16}{3}\cos\varphi + 2\cos^2\varphi \right]_0^{\frac{\pi}{3}} \\
&= 2\pi \left(\left(-\frac{8}{3} + \frac{1}{2} \right) - \left(-\frac{16}{3} + 2 \right) \right) = 2\pi \frac{7}{6} \\
&= \frac{7\pi}{3}.
\end{aligned}
$$

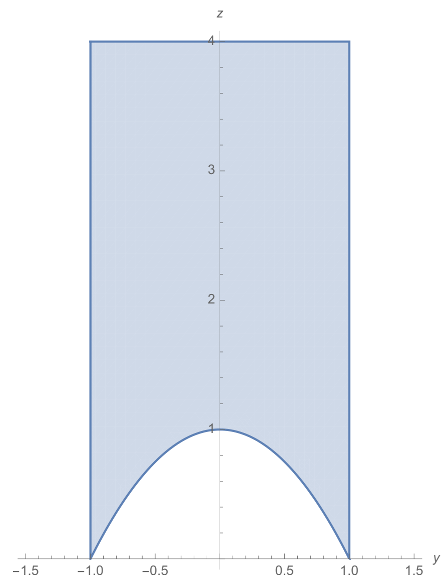
As before, we may make use of symmetry to argue that $x_c = 0$ and $y_c = 0$. The z-coordinate of the centre of gravity is computed as

$$
\begin{aligned}
z_c &= \frac{1}{M(\Omega)} z(2-z)\,dV \\
&= \frac{3}{7\pi} \int_0^{2\pi} \left(\int_0^{\frac{\pi}{3}} \left(\int_0^2 (r\cos\varphi)(2 - r\cos\varphi)(r^2\sin\varphi)\,dr \right) d\varphi \right) d\theta \\
&= \frac{3}{7\pi} \int_0^{2\pi} \left(\int_0^{\frac{\pi}{3}} \left(\int_0^2 (2r^3\cos\varphi\sin\varphi - r^4\cos^2\varphi\sin\varphi)\,dr \right) d\varphi \right) d\theta \\
&= \frac{3}{7\pi} \left(\int_0^{\frac{\pi}{3}} \left(\frac{r^4}{2}\cos\varphi\sin\varphi - \frac{r^5}{5}\cos^2\varphi\sin\varphi \right) \Big|_{r=0}^{2} d\varphi \right) \left(\int_0^{2\pi} d\theta \right) \\
&= \frac{3}{7\pi} \left(\int_0^{\frac{\pi}{3}} \left(8\cos\varphi\sin\varphi - \frac{32}{5}\cos^2\varphi\sin\varphi \right) d\varphi \right) (2\pi) \\
&= \frac{6}{7} \left(-4\cos^2\varphi + \frac{32\cos^3\varphi}{15} \right) \Big|_{\varphi=0}^{\frac{\pi}{3}} \\
&= \frac{6}{7} \left(\left(-1 + \frac{4}{15} \right) - \left(-4 + \frac{32}{15} \right) \right) = \frac{6}{7}\frac{17}{15} = \frac{34}{35}.
\end{aligned}
$$

(d) Let Ω be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the total mass and centre of gravity of Ω.

Solution. The region can be visualized as in the following figures. The second figure shows the region on the yz-axis.





The density function is $\sigma(x, y, z) = k\sqrt{x^2 + y^2} = kr$, where k is some constant. (This in the function in $\mathbb{R}^3$ that is proportional to the distance from the z-axis). The mass can be computed using cylindrical coordinates as

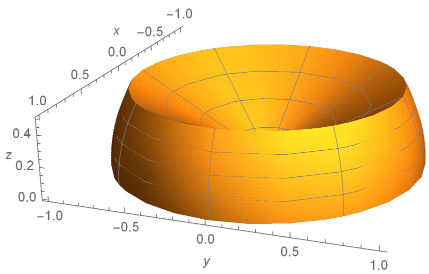
$$
\begin{aligned}
M(\Omega) &= \iiint_\Omega k\sqrt{x^2 + y^2}\, dV \\
&= \left(\int_0^{2\pi} d\theta\right) \int_0^{2\pi} \left(\int_0^1 \left(\int_{1-r^2}^4 (kr)(r)\, dz\right) dr\right) d\theta \\
&= k\left(\int_0^1 r^2 \int_{1-r^2}^4 dz\, dr\right) \\
&= 2k\pi \int_0^1 r^2(4 - 1 + r^2) dr = 2k\pi \int_0^1 (r^4 + 3r^2) dr \\
&= 2k\pi\left(\frac{1}{5} + 1\right) = \frac{12k\pi}{5}.
\end{aligned}
$$

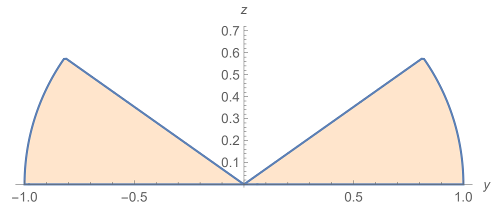
By rotational symmetry, the center of gravity of the ice-cream cone will have $x_c = 0$ and $y_c = 0$. The z-coordinate for the centre of gravity is

$$
z_c = \frac{1}{M(\Omega)} \iiint_\Omega (z)(k\sqrt{x^2 + y^2})\, dV = \frac{5}{12\pi} \int_0^{2\pi} \left(\int_0^1 \left(\int_{1-r^2}^4 rrz\, dz\right) dr\right) d\theta = \frac{46}{21}.
$$

2. Find the volume of the the region inside the unit hemisphere $z = \sqrt{1 - x^2 - y^2}$ but **outside** the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$.

   Solution. The region can be visualized as shown in the following figures. The second figure shows the region on the plane $x = 0$ (i.e., in the yz-plane).





The angle that the cone makes with the z-axis is $\frac{\pi}{3}$. We can make use of spherical coordinates to compute the integral, where the region can be parameterized by $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, and $z = \rho\cos\varphi$ for parameters in the ranges

$$0 \le \rho \le 1, \qquad 0 \le \theta \le 2\pi, \qquad \frac{\pi}{2} \le \varphi \le \frac{\pi}{3}.$$

The volume may be computed as

$$\begin{aligned}
\iiint_\Omega dV &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 (\rho^2 \sin\varphi)\, d\rho\, d\theta\, d\varphi \\
&= \left(\int_0^1 \rho^2\, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\, d\varphi\right) \\
&= \left(\frac{\rho^3}{3}\bigg|_0^1\right)(2\pi)\left(-\cos\varphi\bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right) \\
&= \frac{2\pi}{3}\left(0 - \left(-\frac{1}{2}\right)\right) = \frac{\pi}{3}.
\end{aligned}$$

3. Determine the curl and divergence of the vector field defined for all $\boldsymbol{r} \neq \boldsymbol{0}$ as $\boldsymbol{F}(\boldsymbol{r}) = \dfrac{\boldsymbol{r}}{r^p}$, where $\boldsymbol{r} = (x, y, z)$ and $r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, and $p > 0$ is a constant.

Solution. Here we will make use of the following two vector calculus identities. If $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and $g : \mathbb{R}^3 \to \mathbb{R}$ is a function, we have

$$\nabla \cdot (g\boldsymbol{G}) = (\nabla g) \cdot \boldsymbol{G} + g(\nabla \cdot \boldsymbol{G}) \quad \text{and} \quad \nabla \times (g\boldsymbol{G}) = \nabla g \times \boldsymbol{G} + g(\nabla \times \boldsymbol{G}).$$

We may write the field as a product

$$\boldsymbol{F}(\boldsymbol{r}) = \left(\frac{1}{r^p}\right)\boldsymbol{r}$$

and use the first identity to find that

$$\begin{aligned}
\nabla \cdot \boldsymbol{F} = \nabla \cdot \left(\left(\frac{1}{r^p}\right)\boldsymbol{r}\right) &= \nabla\left(\frac{1}{r^p}\right) \cdot \boldsymbol{r} + \left(\frac{1}{r^p}\right)\nabla \cdot \boldsymbol{r} \\
&= -\frac{p}{r^{p+1}}\nabla r \cdot \boldsymbol{r} + \frac{3}{r^p}
\end{aligned}$$

using the chain rule and the fact that $\boldsymbol{r} = (x, y, z)$. More specifically, the chain rule applied here yields that

$$\nabla\left(\frac{1}{r^p}\right) = \left(\frac{d}{dr}\left(\frac{1}{r^p}\right)\right)\nabla r.$$

Moreover, note that

$$\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\boldsymbol{r}}{r}.$$

We therefore have that

$$\nabla\cdot\boldsymbol{F} = -\frac{p}{r^{p+2}}\boldsymbol{r}\cdot\boldsymbol{r} + \frac{3}{r^p} = -\frac{pr^2}{r^{p+2}} + \frac{3}{r^p} = \frac{3-p}{r^p}$$

where we use the fact that $\boldsymbol{r}\cdot\boldsymbol{r} = r^2$. Using the second identity, we have that

$$\begin{aligned}\nabla\times\boldsymbol{F} = \nabla\times\left(\frac{1}{r^p}\right)\boldsymbol{r} &= \nabla\left(\frac{1}{r^p}\right)\times\boldsymbol{r} + \left(\frac{1}{r^p}\right)\nabla\times\boldsymbol{r} \\ &= \left(-\frac{p}{r^{p+1}}\right)\nabla r\times\boldsymbol{r} + \boldsymbol{0} = \left(-\frac{p}{r^{p+2}}\right)\boldsymbol{r}\times\boldsymbol{r} + \boldsymbol{0} = \boldsymbol{0}\end{aligned}$$

where we have used the calculation from above for ∇r and the fact that $\nabla\times\boldsymbol{r} = \boldsymbol{0}$ and $\boldsymbol{r}\times\boldsymbol{r} = \boldsymbol{0}$.

<u>Comment</u>: There are a couple of things to notice. First, this is a radial vector field in which the field lines emanate from the origin symmetrically outward into space. The fact that the vector field is irrotational (i.e. the curl is zero) should make intuitive sense.
When $p = 3$, we see also that $\nabla\cdot\boldsymbol{F} = 0$. Examples of radial vector fields with $p = 3$ include the gravitational field and the electric field induced by a point charge. Such vector fields are called *solenoidal* (or incompressible).

4. A repeat from the last set of practice problems: Let $b, k, \ell > 0$ be positive constants and consider the region defined by $\Omega = \{(x,y,z) : x^2+z^2 \le b^2 \text{ and } -\ell \le y \le \ell\}$ that is inside the cylinder $\{(x,y,z) : x^2+z^2 = b^2 \text{ and } -\ell \le y \le \ell\}$.

   (a) Let a fluid flow have velocity $\boldsymbol{v} = (0,0,kz)$ and constant density ρ_0. Compute the integral $\iiint_\Omega \nabla\cdot\boldsymbol{F}\,dV$, where $\boldsymbol{F} = \rho_0\boldsymbol{v}$ is the mass flux vector.

   Solution. The divergence of the mass flux is $\nabla\cdot\boldsymbol{F} = \rho_0 k$. We therefore have that

   $$\iiint_\Omega \nabla\cdot\boldsymbol{F}\,dV = \iiint_\Omega \rho_0 k\,dV = \rho_0 k\iiint_\Omega dV = 2\pi\rho_0 k b^2\ell,$$

   where we have used the volume of the cylinder, which is $\pi b^2(2\ell)$.

   (b) Compute $\iiint_\Omega \nabla\cdot\boldsymbol{F}\,dV$, this time with $\boldsymbol{v} = (kx, ky, kz)$.

   Solution. In this case, the divergence is $\nabla\cdot\boldsymbol{F} = 3\rho_0 k$ which yields

   $$\iiint_\Omega \nabla\cdot\boldsymbol{F}\,dV = 6\pi\rho_0 k b^2\ell.$$

   (c) Did you get the same answers as those in the last assignment for the flux? Explain why or why not, and verify Gauss' Theorem by completing any necessary calculations.

Solution. We obtained the same in part (a), but a different answer in part (b). The reason is that when we computed the flux in last week's problems, we only considered the flux through the cylinder. The cylinder is *not closed*, so Gauss' theorem does not apply. In order to use Gauss' theorem, the ends of the cylinder must be considered as well. The ends of the cylinder are the disks $x^2 + z^2 = b^2$ at the planes $y = \pm\ell$. These surfaces have normal vectors given by $\hat{\boldsymbol{n}} = (0, \pm 1, 0)$.

In part (a), for both ends of the cylinder we have that $\boldsymbol{F} \cdot \hat{\boldsymbol{n}} = 0$, and therefore there is no flux through the ends. This makes sense intuitively from the form of the vector field.

In part (b), on the end of the cylinder with $y = \ell$, the outward unit normal vector is $\hat{\boldsymbol{n}} = (0, 1, 0)$. We will call this surface $\Sigma_{y=\ell}$. On this face, we have that $\boldsymbol{F} \cdot \hat{\boldsymbol{n}} = \rho_0 k v = \rho_0 k \ell$. The flux through this end is therefore given by

$$\iint_{\Sigma_{y=\ell}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA = \rho_0 k \ell \iint_{\Sigma_{y=\ell}} dA = \pi \rho_0 k b^2 \ell$$

since the area of the face is πb^2.

On the end of the cylinder with $y = -\ell$, the outward normal is $\hat{\boldsymbol{n}} = (0, -1, 0)$. We can call this surface $\Sigma_{y=-\ell}$. On this face we have that $\boldsymbol{F} \cdot \hat{\boldsymbol{n}} = -\rho_0 k v = -\rho_0 k(-\ell)$. The flux is again $\pi \rho_0 k b^2 \ell$.

In the last assignment, we found the flux from the cylinder was $4\pi \rho_0 k b^2 \ell$. Summing this flux with that over the two ends, the same answer is obtained as using the Divergence Theorem.

5. For the following vector fields $\boldsymbol{F}$, determine whether or not they are solenoidal and find a corresponding vector potential field $\boldsymbol{G}$ if it exists.

(a) $\boldsymbol{F}(x, y, z) = (yz, xz, xy)$

Solution. Note that $\nabla \cdot \boldsymbol{F} = 0 + 0 + 0 = 0$, so a vector potential $\boldsymbol{G}$ exists. We may assume that it has the form $\boldsymbol{G} = (G_1, G_2, 0)$, where G_1 and G_2 satisfy the differential equations

$$\begin{aligned} -\frac{\partial G_2}{\partial z} &= yz \\ \frac{\partial G_1}{\partial z} &= xz \\ \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= xy. \end{aligned}$$

Integrating the first two equations with respect to z, we find that

$$G_2(x, y, z) = -\frac{1}{2} y z^2 + f(x, y) \qquad \text{and} \qquad G_1(x, y, z) = \frac{1}{2} x z^2 + g(x, y),$$

where f and g are functions that are independent of z. Taking the derivatives of G_2 and G_1 with respect to x and y, we find that

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial f}{\partial x}(x, y) - \frac{\partial g}{\partial y}(x, y) = xy.$$

We may take the solution $g(x,y) = 0$ and $f(x,y) = \frac{1}{2}x^2y$ such that a vector potential field for $\boldsymbol{F}$ may be given as

$$\boldsymbol{G}(x,y,z) = \frac{1}{2}\left(xz^2, x^2y - yz^2, 0\right).$$

(b) $\boldsymbol{F}(x,y,z) = (xz^2 - 1,\ -yz^2,\ 1 - x^2)$

Solution. Note that $\nabla \cdot \boldsymbol{F} = z^2 - z^2 + 0 = 0$, so a vector potential $\boldsymbol{G}$ exists. We may assume that it has the form $\boldsymbol{G} = (G_1, G_2, 0)$. Following the same method as above, we find that G_1 and G_2 must be of the form

$$G_2(x,y,z) = -\frac{1}{3}xz^3 + z + f(x,y) \qquad \text{and} \qquad G_1(x,y,z) = -\frac{1}{3}yz^3 + g(x,y),$$

where f and g are functions that are independent of z. Taking the derivatives of G_2 and G_1 with respect to x and y, we find that

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = -\frac{1}{3}z^3 + \frac{\partial f}{\partial x}(x,y) + \frac{1}{3}z^3 - \frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial x}(x,y) - \frac{\partial g}{\partial y}(x,y) = 1 - x^2.$$

We may take the solution $f(x,y) = 0$ and $g(x,y) = y(x^2 - 1)$ such that a vector potential field for $\boldsymbol{F}$ may be given as

$$\boldsymbol{G}(x,y,z) = \left(y(x^2-1) - \frac{yz^3}{3},\ z - \frac{xz^3}{3},\ 0\right).$$

(c) $\boldsymbol{F}(x,y,z) = (x,y,z)$

Solution. Note that $\nabla \cdot \boldsymbol{F} = 1 + 1 + 1 = 3 \neq 0$, so $\boldsymbol{F}$ is not solenoidal and thus no vector potential field can exist.

6. Find the flux of the vector field $\boldsymbol{F}(x,y,z) = (xz, -yz, 1 + y^2)$ across the surface Σ that is defined as $\Sigma = \{(x,y,z) \,|\, z = \cos^{-1}(x^2+y^2) \text{ and } x^2 + y^2 \leq 1\}$ with outward pointing normal.

   Hint: Do not attempt to do this directly. Instead, notice that $\nabla \cdot \boldsymbol{F} = 0$ which implies there is a vector potential, call it $\boldsymbol{G}$, such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$. Also, note that vector potentials are not unique, and therefore you can choose any $\boldsymbol{G}$ that satisfies the above.

Solution. We first notice that $\nabla \cdot \boldsymbol{F} = z - z = 0$, and thus the vector field $\boldsymbol{F}$ is solenoidal. We will attempt to construct a vector field $\boldsymbol{G}$ with components $\boldsymbol{G}(x,y,z) = (G_1(x,y,z), G_2(x,y,z), G_3(x,y,z))$ that is a vector potential for $\boldsymbol{F}$.

We first start with the F_1 and F_2 components to see if we can find G_3 that works. The components of $\boldsymbol{F}$ must satisfy $F_1 = \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = xz$ and $F_2 = \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = -yz$. By inspection, notice that if $G_3 = xyz$, then $\frac{\partial G_2}{\partial z} = 0$ and $\frac{\partial G_1}{\partial z} = 0$ satisfies the requirement. Thus we may conclude that $G_1 = G_1(x,y)$ and $G_2 = G_2(x,y)$ are functions of x and y only.

Now look at $F_3 = \dfrac{\partial G_2}{\partial x} - \dfrac{\partial G_1}{\partial y} = 1 + y^2$. If we choose $G_2 = x$ and $G_1 = -\dfrac{y^3}{3}$, this equation is satisfied. We could have alternatively chosen $G_2 = xy^2$ and $G_1 = -y$.

However, if possible we choose $\boldsymbol{G}$ so that one component is zero, making the calculation easier later on. So we'll choose $G_2 = x(1+y^2)$ and $G_1 = 0$. It follows that $\boldsymbol{G} = \big(0, x(1+y^2), xyz\big)$ is a vector potential for $\boldsymbol{F}$.

We have that

$$\iint_\Sigma \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dA = \iint_\Sigma (\nabla\times\boldsymbol{G})\cdot\hat{\boldsymbol{n}}\,dA = \int_{\partial\Sigma} \boldsymbol{G}\cdot\,d\boldsymbol{r}$$

by Stokes' Theorem. The boundary curve $\partial\Sigma$ is defined as $x^2+y^2=1$ with $z=0$, which can be parameterized by $\boldsymbol{\gamma}(t) = (\cos t, \sin t, 0)$ for $t\in[0,2\pi]$. Then

$$\int_{\partial\Sigma} \boldsymbol{G}\cdot\,d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{G}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt = \int_0^{2\pi} \cos^2 t(1+\sin^2 t)\,dt = \cdots = \frac{5\pi}{4}$$

where the half angle identities have been applied and we note that integrating a function such as $\cos(2t)$ over $[0,2\pi]$ will result in a value of zero.

7. Derive the following vector calculus identity. For a vector field $\boldsymbol{F}$, it holds that

$$\nabla\times(\nabla\times\boldsymbol{F}) = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

where $\nabla^2\boldsymbol{F}$ is the *vector Laplacian* of $\boldsymbol{F}$ defined as $\nabla^2\boldsymbol{F} = \big(\nabla^2F_1, \nabla^2F_2, \nabla^2F_3\big)$.

Solution. We must assume that $\boldsymbol{F}$ is $\mathcal{C}^2$ (i.e., has continuous 2nd order partial derivatives). Let $\boldsymbol{F} = (F_1, F_2, F_3)$. Recall that the curl of $\boldsymbol{F}$ is

$$\nabla\times\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right),$$

and so the curl of the curl of $\boldsymbol{F}$ is

$$\begin{aligned}
\nabla\times(\nabla\times\boldsymbol{F}) &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix} \\
&= \left(\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y\partial y} - \frac{\partial^2 F_1}{\partial z\partial z} + \frac{\partial^2 F_3}{\partial z\partial x}\right)\hat{\boldsymbol{\imath}} \\
&\quad + \left(\frac{\partial^2 F_3}{\partial z\partial y} - \frac{\partial^2 F_2}{\partial z\partial z} - \frac{\partial^2 F_2}{\partial x\partial x} + \frac{\partial^2 F_1}{\partial x\partial y}\right)\hat{\boldsymbol{\jmath}} \\
&\quad + \left(\frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x\partial x} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial y\partial 2}\right)\hat{\boldsymbol{k}} \\
&= \left(\frac{\partial^2 F_1}{\partial x\partial x} + \frac{\partial^2 F_2}{\partial y\partial x} + \frac{\partial^2 F_3}{\partial z\partial x}\right)\hat{\boldsymbol{\imath}} - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{\boldsymbol{\imath}} \\
&\quad + \left(\frac{\partial^2 F_1}{\partial x\partial y} + \frac{\partial^2 F_2}{\partial y\partial y} + \frac{\partial^2 F_3}{\partial z\partial y}\right)\hat{\boldsymbol{\jmath}} - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{\boldsymbol{\jmath}} \\
&\quad + \left(\frac{\partial^2 F_1}{\partial x\partial z} + \frac{\partial^2 F_2}{\partial y\partial z} + \frac{\partial^2 F_3}{\partial z\partial z}\right)\hat{\boldsymbol{k}} - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\hat{\boldsymbol{k}} \\
&= \frac{\partial}{\partial x}(\nabla\cdot F)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}(\nabla\cdot F)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}(\nabla\cdot F)\hat{\boldsymbol{k}} - \nabla^2\boldsymbol{F},
\end{aligned}$$

as desired, where we recall that $\nabla\cdot\boldsymbol{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ and we may switch the order of the partial derivatives since $\boldsymbol{F}$ is $\mathcal{C}^2$.

8. Gauss' Law: The electric field $\boldsymbol{E} : \{\boldsymbol{r} \in \mathbb{R}^3 : \boldsymbol{r} \neq \boldsymbol{0}\} \to \mathbb{R}$ due to a point charge Q at the origin is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{kQ}{r^3}\boldsymbol{r}$$

where $\boldsymbol{r} = (x, y, z)$, $r = \|\boldsymbol{r}\|$, and $k = \dfrac{1}{4\pi\epsilon_0}$ is constant. Show that

$$\oiint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA = \begin{cases} 0 & \text{if } \Sigma \text{ does not enclose the charge} \\ \dfrac{Q}{\epsilon_0} & \text{if } \Sigma \text{ encloses the charge} \end{cases}$$

where Σ is an arbitrary smooth, closed surface. For the second case, you may assume that Σ is a sphere centered at the origin.

Solution. Let Σ be an arbitrary smooth, closed surface in $\mathbb{R}^3$. First suppose that Σ does not enclose the charge (i.e. Σ does not enclose the origin, which is the point $\boldsymbol{r} = \boldsymbol{0}$). Then $\boldsymbol{E}$ is defined everywhere inside the region contained in Σ, and the divergence theorem applies. Let Ω be the region enclosed by Σ such that $\partial\Omega = \Sigma$. We have that

$$\iint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA = \iiint_\Omega \nabla \cdot \boldsymbol{E}\, dV = \iiint_\Omega \nabla \cdot \left(\frac{q}{r^3}\boldsymbol{r}\right)\, dV = q \iiint_\Omega \nabla \cdot \left(\frac{\boldsymbol{r}}{r^3}\right)\, dV = 0$$

where the last step follows from the fact that $\nabla \cdot \left(\frac{\boldsymbol{r}}{r^3}\right) = 0$ for all $\boldsymbol{r} \neq \boldsymbol{0}$ (shown in Problem 3).

Now, let Σ be a sphere of radius a centered at the origin. Since $\boldsymbol{E}$ is not defined everywhere inside the region enclosed by Σ (i.e. it is not defined at the origin), the divergence theorem is *not* applicable. Moreover, it holds that the unit normal vector on the surface of the sphere of radius a is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{r}}{a},$$

It follows that

$$\iint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA = \iint_\Sigma \left(\frac{q}{a^3}\boldsymbol{r}\right) \cdot \left(\frac{\boldsymbol{r}}{a}\right)\, dA = \frac{q}{a^4} \iint_\Sigma \boldsymbol{r} \cdot \boldsymbol{r}\, dA = \frac{q}{a^2} \iint_\Sigma dA = \frac{q}{a^2}(4\pi a^2) = \frac{Q}{\epsilon_0},$$

where we use the facts that $\boldsymbol{r} \cdot \boldsymbol{r} = r^2 = a^2$ on the sphere and the surface area of the sphere is equal to $4\pi a^2$.

Note: In this very special case looking at the flux of a radial vector field through a sphere, it turns out parameterization is not necessary, as demonstrated above.

9. Consider the cube $\{(x, y, z) \mid -1 \leq x, y, z \leq 1\}$. Find the charge enclosed by the cube if the electric field is:

(a) $\boldsymbol{E}(x, y, z) = (x, y, z)$

Solution. First, we require Gauss' Law, which states that the net charge enclosed by a closed surface Σ is

$$Q = \epsilon_0 \iint_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{A}$$

where ϵ_0 is the permittivity of free space.
First consider the face on the plane $x = 1$. This is the square with $-1 \leq y \leq 1$ and $-1 \leq z \leq 1$. The unit normal to the surface is $\boldsymbol{n} = \hat{\boldsymbol{\imath}} = (1, 0, 0)$. The vector field is $\boldsymbol{E} = (1, y, z)$. Therefore, we have $\boldsymbol{E} \cdot \boldsymbol{n} = 1$. The flux through this face is therefore given by

$$\iint_{D_{x=1}} \boldsymbol{E} \cdot \boldsymbol{n}\, dy\, dz = \iint_{D_{x=1}} dy\, dz = 4,$$

which is just the area of the face.
Now consider $x = -1$. We now have $\boldsymbol{n} = -\hat{\boldsymbol{\imath}} = (-1, 0, 0)$ and $\boldsymbol{E} = (-1, y, z)$ so that $\boldsymbol{E} \cdot \boldsymbol{n} = 1$. As before, we have that the flux through this face is given by

$$\iint_{D_{x=-1}} \boldsymbol{E} \cdot \boldsymbol{n}\, dy\, dz = \iint_{D_{x=1}} dy\, dz = 4.$$

By symmetry, the same thing will happen for the $y = \pm 1$ and $z = \pm 1$ faces, so the total flux will be $4 \times 6 = 24$ (since there are six faces, each contributing a flux of 4 to the total flux). Therefore, the total charge enclosed inside the sphere is equal to $Q = 24\epsilon_0$.

(b) $\boldsymbol{E}(x, y, z) = (x^2, y^2, z^2)$

Solution. We first examine the $x = \pm 1$ faces again. Through the face on the plane $x = 1$, we have that the normal vector is $\boldsymbol{n} = (1, 0, 0)$ and the field is $\boldsymbol{E} = (1, y^2, z^2)$, so the flux through this face is equal to 4, as in part (a). For the face on the plane $x = -1$, however, the normal vector is again $\boldsymbol{n} = (-1, 0, 0)$ but the field is given by $\boldsymbol{E} = (1, y^2, z^2)$, so $\boldsymbol{E} \cdot \boldsymbol{n} = -1$ which implies the flux through this face is -4. Thus, the positive and negative contributions on each pair of faces will cancel one another out, and the flux will be zero. Therefore, $Q = 0$.

10. For each law below, write the law mathematically in terms of integrals. Then use theorems from vector calculus to derive a partial differential equation that holds at every point in space, each leading to Maxwell's equations in differential form.

(a) Faraday's Law: *The circulation of an electric field* $\boldsymbol{E}$ *around the perimeter of a surface is equal to the negative time rate of change of the flux of the magnetic field* $\boldsymbol{B}$ *through the surface.*

Solution. Let Σ represent a surface and $\partial\Sigma$ be its boundary curve. The law states that

$$\oint_{\partial\Sigma} \boldsymbol{E} \cdot\, d\boldsymbol{r} = -\frac{\partial}{\partial t} \iint_{\Sigma} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, dA.$$

Start by applying Stokes' Theorem on the left hand side, and bringing the time derivative inside the integral on the right to find that

$$\iint_{\Sigma} (\nabla \times \boldsymbol{E}) \cdot \hat{\boldsymbol{n}}\, dA = -\iint_{\Sigma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{\boldsymbol{n}}\, dA,$$

from which we conclude that

$$\iint_{\Sigma}\left(\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}\right)\cdot\hat{\boldsymbol{n}}\,dA=0$$

Since the surface Σ is arbitrary, by the du Bois-Reymond lemma, we obtain Maxwell's second equation relating the electric and magnetic fields, which states that

$$\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=\mathbf{0}\quad\text{or}\quad\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}.$$

(b) No magnetic monopoles: *The flux of the magnetic field through any closed surface is zero.*

Solution. Let $\Omega\subseteq\mathbb{R}^3$ be any region, and let $\partial\Omega$ be the closed surface which encloses Ω. In integral form, the law states that

$$\iint_{\partial\Omega}\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,dA=0.$$

Using the Divergence Theorem on the left, we have that

$$\iiint_{\Omega}\nabla\cdot\boldsymbol{B}\,dV=0.$$

This integral must vanish for *any* possible choice of region Ω, which can only happen if the integrand itself is equal zero (by the du Bois-Reymond Lemma). This implies that $\nabla\cdot\boldsymbol{B}=0$, as desired.

(c) Ampere's Law: *The circulation of the magnetic field around the perimeter of a surface is equal to the time rate of change of the flux of the electric field through the surface + the flux of the electric current density through the surface* (where there are constants of proportionality).

Solution. Let Σ represent a surface and $\partial\Sigma$ be its boundary curve. The integral form of the law is stated as

$$\oint_{\partial\Sigma}\boldsymbol{B}\cdot d\boldsymbol{r}=\mu_0\epsilon_0\frac{\partial}{\partial t}\iint_{\Sigma}\boldsymbol{E}\cdot\hat{\boldsymbol{n}}\,dA+\mu_0\iint_{\Sigma}\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,dA$$

Applying Stokes' theorem on the right, we have

$$\oint_{\partial\Sigma}\boldsymbol{B}\cdot d\boldsymbol{r}=\iint_{\Sigma}(\nabla\times\boldsymbol{B})\cdot\hat{\boldsymbol{n}}\,dA$$

Bringing the time derivative inside the integral on the right and re-arranging, we obtain

$$\iint_{\Sigma}\left[\nabla\times\boldsymbol{B}-\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}-\mu_0\boldsymbol{J}\right]\cdot\hat{\boldsymbol{n}}\,dA=0,$$

which by the du Bois-Reymond Lemma gives the following PDE:

$$\nabla\times\boldsymbol{B}-\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}-\mu_0\boldsymbol{J}=\mathbf{0}$$

or, equivalently,

$$\nabla\times\boldsymbol{B}=\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}+\mu_0\boldsymbol{J}.$$

Vector calculus is useful for all sorts of engineering and physics applications, not just electromagnetism. In particular, heat flow and fluid flow are modeled well by vector calculus. We will not go into these examples in any more detail in the course, but these problems are here to show you some of the further physical applications of vector calculus.

1. A volume Ω of a homogeneous and isotropic material is bounded by a smooth orientable surface $\partial\Omega$ and is being heated from outside. Since heat is energy and energy is conserved, it will be true that:

$$(\text{increase of heat in } \Omega) = (\text{flux of heat through } \partial\Omega) \tag{1}$$

Let the heat flux vector be denoted by $\boldsymbol{J}(\boldsymbol{r}, t)$, and the heat energy density be given by $\rho C_p T(\boldsymbol{r}, t)$, where ρ is the mass density, C_p is the heat capacity (both constant) and $T(\boldsymbol{r}, t)$ the temperature.

(a) Using an appropriate theorem of integral vector calculus, translate (1) into a mathematical statement.

Solution. Since $\rho C_p T(\boldsymbol{r}, t)$ is the heat energy density, the total heat energy is given by the triple integral, and thus the rate of change is:

$$\frac{d}{dt}\iiint_\Omega \rho C_p T \, dV = \iiint_\Omega \rho C_p \frac{\partial T}{\partial t} \, dV$$

where the t derivative can be brought inside the integral since Ω does not depend on t.
The heat flux through $\partial\Omega$ is $-\iint_{\partial\Omega} \boldsymbol{J} \cdot \hat{\boldsymbol{n}} \, dA$ (negative because being heated from outside).
Using the divergence theorem this can be expressed:

$$\iint_{\partial\Omega} \boldsymbol{J} \cdot \hat{\boldsymbol{n}} \, dA = \iiint_\Omega \nabla \cdot \boldsymbol{J} \, dV$$

Equating the expressions:

$$\iiint_\Omega \rho C_p \frac{\partial T}{\partial t} = -\iiint_\Omega \nabla \cdot \boldsymbol{J} \, dV \Longrightarrow \iiint_\Omega \left(\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{J}\right) = 0$$

Since Ω is arbitrary and the integrand is continuous (by the DuBois-Raymond Lemma), we get:

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

PDE. Try it.

(b) Fourier's heat says that $\boldsymbol{J} = -k\nabla T$ where the constant k represents the *thermal conductivity* of the material. Use this to derive a partial differential equation that $T(\boldsymbol{r}, t)$ must satisfy.

Solution. Substituting into part (a), we get

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot (-k\nabla T) = 0 \Longrightarrow \frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where $\alpha = \dfrac{k}{\rho C_p}$ is known as the *thermal diffusivity*.
This is called the *Heat equation* or *Diffusion equation*.
Remark: This derivation makes the explicit assumption that the thermal conductivity k is constant. This is approximately true in many cases, but in real-life problems k could be

a function of temperature and other variables, which makes the expansion in the last step become a bit more complicated. The result: a non-linear PDE. Try it.

2. Consider a region Ω of a homogeneous solution with boundary $\partial\Omega$ (a smooth orientable surface). Suppose that a substance M is dissolved in the solution. Since mass is conserved, it will be true that:

$$(\text{rate of change of amount of } M \text{ in } \Omega) = -\,(\text{flux of } M \text{ through } \partial\Omega) \tag{2}$$

where the negative sign appears because a net flux outward corresponds to a decrease in the amount of M. Let the flux vector for M be denoted by $\boldsymbol{J}(\boldsymbol{r}, t)$, and let the concentration of M be $c(\boldsymbol{r}, t)$.

(a) Using an appropriate theorem of integral vector calculus, translate (2) into a mathematical statement.

Solution. Since $(\boldsymbol{r}, t)$ is the concentration, the total amount of A is given by the triple integral, and thus the rate of change is:

$$\frac{d}{dt}\iiint_\Omega c(\boldsymbol{r}, t)\, dV = \iiint_\Omega \frac{\partial c}{\partial t}\, dV$$

where the t derivative can be brought inside the integral since Ω does not depend on t. The flux of M through $\partial\Omega$ is $-\displaystyle\iint_{\partial\Omega} \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\, dA$. Using the divergence theorem this can be expressed:

$$\iint_{\partial\Omega} \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\, dA = \iiint_\Omega \nabla\cdot\boldsymbol{J}\, dV$$

Equating the expressions:

$$\iiint_\Omega \frac{\partial c}{\partial t}\, dV = -\iiint_\Omega \nabla\cdot\boldsymbol{J}\, dV \Longrightarrow \iiint_\Omega \left(\frac{\partial c}{\partial t} + \nabla\cdot\boldsymbol{J}\right) dV = 0$$

Since Ω is arbitrary and the integrand is continuous (by the DuBois-Raymond Lemma), we get:

$$\frac{\partial c}{\partial t} + \nabla\cdot\boldsymbol{J} = 0$$

(b) *Fick's Law* says that $\boldsymbol{J} = -D\nabla c$, where the constant of D is the *diffusion coefficient*. In words, this is "the flux of M is in the direction of $-\nabla c$, and its magnitude is proportional to $-\nabla c$." Use Fick's law to derive a partial differential equation that $c(\boldsymbol{r}, t)$ must satisfy.

Solution. Substituting into part (a), we get

$$\frac{\partial c}{\partial t} + \nabla\cdot(-D\nabla c) = 0 \Longrightarrow \frac{\partial c}{\partial t} = D\nabla^2 c$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the *Laplacian* operator.
This partial differential equation (PDE) is known as the *diffusion equation.*
Note: It has been assumed that D is constant. If D is a function of c or $\boldsymbol{r}$, for example, the last step of the derivation becomes more complicated and the result is a non-linear PDE. (Try it.)
Question to ponder: In the presence of a flow, there is also a contribution to the flux

from convection. What happens to the PDE in this case?