ECE 206 Fall 2019
Practice Problems Week 8

1 Electromagnetism

1. In class, we derived the wave equation for electric and magnetic fields in a vacuum ($\boldsymbol{J} = \boldsymbol{0}$ and $\rho = 0$) from Maxwell's equations. Here you will derive the inhomogeneous wave equation. Suppose that the charge density $\rho(\boldsymbol{r}, t)$ and current density $\boldsymbol{J}(\boldsymbol{r}, t)$ are both nonzero. Show that the electric field obeys the inhomogeneous wave equation

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{E} = \boldsymbol{F}(\boldsymbol{r}, t)$$

   where $\boldsymbol{F}(\boldsymbol{r}, t)$ is a vector field given in terms of ρ and $\boldsymbol{J}$. Find an expression for $\boldsymbol{F}$. Show that the magnetic field $\boldsymbol{B}$ also obeys an inhomogeneous wave equation.

2. Let $\boldsymbol{B} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as

$$\boldsymbol{B}(x, y, z) = f(x^2 + y^2 + z^2)(y, -x, 0)$$

   where f is an arbitrary C^1 function. Show that $\boldsymbol{B}$ can be a valid magnetostatic field and find the corresponding current density $\boldsymbol{J}(x, y, z, t)$. *Hint:* Let $r = \sqrt{x^2 + y^2 + z^2}$.

3. Suppose that a time-varying charge density field $\rho(x, y, z, t)$ and a time-varying current density field $\boldsymbol{J}(x, y, z, t)$ cause time-varying electric and magnetic fields $\boldsymbol{E}(x, y, z, t)$ and $\boldsymbol{B}(x, y, z, t)$ in accordance with Maxwell's equations.

   (a) Is the electric field conservative?

   (b) Suppose that $\boldsymbol{A}(x, y, z, t)$ is a magnetic vector potential of $\boldsymbol{B}$. Show that the time-varying vector field $\boldsymbol{F}$ defined by

$$\boldsymbol{F} = \boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}$$

   is conservative.

   (c) If $\Psi(x, y, z, t)$ is a scalar potential for $\boldsymbol{F}$, use Maxwell's equation to show that

$$\nabla^2 \Psi - \frac{\partial(\nabla \cdot \boldsymbol{A})}{\partial t} = \frac{\rho}{\epsilon_0}.$$

4. An electric field is given by $\boldsymbol{E}(t, x, y, z) = \sin(\omega t - kz)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ in the source-free case for which $\rho(t, x, y, z) = 0$, $\boldsymbol{J}(t, x, y, z) = \boldsymbol{0}$, for all (t, x, y, z), where ω is a constant. Let Γ be the circular boundary of a disc of radius a in the xy-plane with centre at the origin and counterclockwise around the z-axis. If $\boldsymbol{B}(t, x, y, z)$ is the corresponding magnetic field, evaluate the circulation of the magnetic field around Γ as a function of t. That is, evaluate the line integral

$$\oint_\Gamma \boldsymbol{B} \cdot d\boldsymbol{r}$$

   as a function of t.

5. The most general form of a *plane wave* is given by

$$\boldsymbol{E}(t, \boldsymbol{r}) = f(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})\boldsymbol{E}_0$$

where $\boldsymbol{k} = (k_1, k_2, k_3)$ is the *wave vector* (which is *not* the same thing as $\hat{\boldsymbol{k}} = (0, 0, 1)$), f is an arbitrary function, $\boldsymbol{E}_0$ is a constant vector, and $\boldsymbol{r} = (x, y, z)$.

(a) Show that this field satisfies the wave equation $\dfrac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\nabla^2\boldsymbol{E} = \boldsymbol{0}$ as long as the dispersion relation $\omega^2 = c^2k^2$ is satisfied, where $k^2 = k_1^2 + k_2^2 + k_3^2 = \|\boldsymbol{k}\|^2$.

(b) Although the wave equation is derived from Maxwell's equations, this does not imply that every solution of the wave equation is a solution of Maxwell's equations. In general, other conditions must be satisfied. What other condition must the plane wave above satisfy in order to be also a solution of Maxwell's equations?

(c) As a further example, show that the $\boldsymbol{E}$-field

$$\boldsymbol{E}(x, y, z, t) = (0, 0, E_0 \cos(\omega t - kz))$$

satisfies the wave equation but not Maxwell's equations.

6. A static electric field $\boldsymbol{E}(x, y, z)$ is caused by the charge distribution $\rho(x, y, z)$. The potential function is given by

$$\Psi(\boldsymbol{r}) = \begin{cases} -\dfrac{\rho_0 R^3}{3\epsilon_0 r}, & r \geq R \\ \dfrac{\rho_0}{6\epsilon_0}r^2 - \dfrac{\rho_0}{2\epsilon_0}R^2, & r < R \end{cases}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) Find the electric field $\boldsymbol{E}(\boldsymbol{r})$.

(b) Find the charge distribution $\rho(x, y, z)$.

(c) Compute the surface integral $\displaystyle\iint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}}\, dA$ where Σ is the sphere $x^2 + y^2 + z^2 = 4R^2$ for $R > 0$ constant.

2 Complex numbers

1. Sketch the set of points in the complex plane satisfying:

(a) $|z| = 1$

(b) $|z - j - 1| = 4$

(c) $1 \leq |2z - 6| \leq 2$

(d) $|z - j|^2 + |z + j|^2 \leq 2$

(e) $|z - 1| < |z|$

(f) $0 < \operatorname{Im}(z) < \pi$

(g) $|\operatorname{Arg}(z)| < \frac{\pi}{4}$

(h) $0 \leq \operatorname{Arg}(z - j - 1) < \frac{\pi}{3}$

2. Let f be a mapping of the complex plane and let $A \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$ where f is defined. The *image* of A under f is the set of values $f(A) = \{f(z) : z \in A\}$. For each set A below, find and sketch the image $f(A)$ under the given mapping f.

(a) the set $A = \{x + jy : 1 < x < 2,\ 1 < y < 3\}$ under the mapping $f(z) = z + 2j$

(b) the set $A = \{x + jy : 1 < x < 2,\ 1 < y < \infty\}$ under the mapping $f(z) = 2jz$

(c) the set $A = \{z : \mathrm{Im}(z) = 1\}$ under the mapping $f(z) = z^2$

(d) the set $A = \{z : \mathrm{Re}(z) = 1\}$ under the mapping $f(z) = z^2$

(e) the set $A = \{x + jy : 1 < x < 2,\ 1 < y < 3\}$ under the mapping $f(z) = z^2$

3. Where are the following functions of a complex variable defined? (i.e. find their domains)

(a) $f(z) = \dfrac{z}{z + \bar{z}}$

(b) $f(z) = \dfrac{1}{4 - |z|^2}$