ECE 206 Fall 2019
Practice Problems Week 8
Solutions

1 Electromagnetism

1. In class, we derived the wave equation for electric and magnetic fields in a vacuum ($\boldsymbol{J} = \boldsymbol{0}$ and $\rho = 0$) from Maxwell's equations. Here you will derive the inhomogeneous wave equation. Suppose that the charge density $\rho(\boldsymbol{r}, t)$ and current density $\boldsymbol{J}(\boldsymbol{r}, t)$ are both nonzero. Show that the electric field obeys the inhomogeneous wave equation

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{E} = \boldsymbol{F}(\boldsymbol{r}, t)$$

where $\boldsymbol{F}(\boldsymbol{r}, t)$ is a vector field given in terms of ρ and $\boldsymbol{J}$. Find an expression for $\boldsymbol{F}$. Show that the magnetic field $\boldsymbol{B}$ also obeys an inhomogeneous wave equation.

Solution. Start as before in the case of no sources. Take the curl of Faraday's law and use appropriate identities:

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{E}) &= \nabla \times (-\frac{\partial \boldsymbol{B}}{\partial t}) && \text{(taking curl)} \\
\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} &= -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) && \text{(identity, switching order)} \\
\nabla \left(\frac{\rho}{\epsilon_0}\right) - \nabla^2 \boldsymbol{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu_0 \frac{\partial \boldsymbol{J}}{\partial t} && \text{(Maxwell's 1st/4th equations)} \\
\frac{1}{\epsilon_0} \nabla \rho - \nabla^2 \boldsymbol{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu_0 \frac{\partial \boldsymbol{J}}{\partial t} && (\epsilon_0 \text{ is constant}) \\
\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{E} &= -\frac{1}{\epsilon_0}(c^2 \nabla \rho + \frac{\partial \boldsymbol{J}}{\partial t}) && (\text{where } c^2 = \frac{1}{\mu_0 \epsilon_0})
\end{aligned}$$

where we can express the right-hand side as $\boldsymbol{F}(\boldsymbol{r}, t)$, a known function. This is the *inhomogeneous wave equation.*
A similar procedure starting with Ampère's Law leads to

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{B}) &= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) + \mu_0 \nabla \times \boldsymbol{J} && \text{(taking curl, switching order)} \\
\nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} &= \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \boldsymbol{B}}{\partial t}\right) + \mu_0 \nabla \times \boldsymbol{J} && \text{(identity, Maxwell's 2nd eqn)} \\
\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{B} &= \frac{1}{\epsilon_0}(\nabla \times \boldsymbol{J}) && (\text{since } \nabla \cdot \boldsymbol{B} = 0)
\end{aligned}$$

Again, the right-hand side is a known function, so this is the inhomogeneous wave equation.

2. Let $\boldsymbol{B} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as

$$\boldsymbol{B}(x, y, z) = f(x^2 + y^2 + z^2)(y, -x, 0)$$

where f is an arbitrary C^1 function. Show that $\boldsymbol{B}$ can be a valid magnetostatic field and find the corresponding current density $\boldsymbol{J}(x, y, z, t)$. *Hint:* Let $r = \sqrt{x^2 + y^2 + z^2}$.

Solution. $\boldsymbol{B}$ must satisfy $\nabla \cdot \boldsymbol{B} = 0$. Using our vector identities and the chain rule:

$$\nabla \cdot \boldsymbol{B} = \nabla \left(f(r^2)(y, -x, 0)\right) = \nabla f \cdot (y, -x, 0) + f \nabla \cdot (y, -x, 0) = f' \nabla r(2r) \cdot (y, -x, 0) + 0$$

Since $\nabla r = \dfrac{\boldsymbol{r}}{r}$ with $\boldsymbol{r} = (x, y, z)$, this simplifies to

$$\nabla \cdot \boldsymbol{B} = 2f'(x, y, z) \cdot (y, -x, 0) = 2f'(xy - yx) = 0$$

So $\boldsymbol{B}$ is a static magnetic field. To find the current density, we have from Ampere's Law (in the steady state) that $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$, or equivalently, $\boldsymbol{J} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B}$. Using the vector product rule for curl and our above calculation for ∇f:

$$\begin{aligned} \nabla \times \boldsymbol{B} &= (\nabla f) \times (y, -x, 0) + f\left(\nabla \times (y, -x, 0)\right) \\ &= 2f' \boldsymbol{r} \times (y, -x, 0) + f(0, 0, -2) \\ &= 2f' \left(xz, yz, -(x^2 + y^2)\right) + f(0, 0, -2) \end{aligned}$$

So

$$\boldsymbol{J} = \frac{1}{\mu_0} \left[2f' \left(xz,\ yz,\ -(x^2 + y^2)\right) + f(0, 0, -2)\right]$$

or

$$\boldsymbol{J} = \frac{1}{\mu_0} \left[\left(2f'xz,\ 2f'yz,\ -2(f'(x^2 + y^2) + f)\right)\right]$$

3. Suppose that a time-varying charge density field $\rho(x, y, z, t)$ and a time-varying current density field $\boldsymbol{J}(x, y, z, t)$ cause time-varying electric and magnetic fields $\boldsymbol{E}(x, y, z, t)$ and $\boldsymbol{B}(x, y, z, t)$ in accordance with Maxwell's equations.

   (a) Is the electric field conservative?

   Solution. By Maxwell's equations, $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$, but $-\dfrac{\partial \boldsymbol{B}}{\partial t} \neq \boldsymbol{0}$ in general. So the electric field is not conservative.

   (b) Suppose that $\boldsymbol{A}(x, y, z, t)$ is a magnetic vector potential of $\boldsymbol{B}$. Show that the time-varying vector field $\boldsymbol{F}$ defined by

$$\boldsymbol{F} = \boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}$$

   is conservative.

   Solution. Note that $\boldsymbol{F}$ is conservative if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$. Take the curl of the equation:

$$\nabla \times \boldsymbol{F} = \nabla \times \boldsymbol{E} + \nabla \times \left(\frac{\partial \boldsymbol{A}}{\partial t}\right) = \nabla \times \boldsymbol{E} + \frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}$$

Hence $\boldsymbol{F}$ is conservative.

(c) If $\Psi(x, y, z, t)$ is a scalar potential for $\boldsymbol{F}$, use Maxwell's equation to show that

$$\nabla^2\Psi - \frac{\partial(\nabla \cdot \boldsymbol{A})}{\partial t} = \frac{\rho}{\epsilon_0}.$$

Solution. Take the divergence of the given equation:

$$\nabla \cdot \boldsymbol{F} = \nabla \cdot \boldsymbol{E} + \nabla \cdot \frac{\partial \boldsymbol{A}}{\partial t}.$$

From Maxwell's first equation, we have that $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon}$. Additionally, we have $\boldsymbol{F} = \nabla\Psi$, so $\nabla \cdot \boldsymbol{F} = \nabla \cdot (\nabla\Psi) = \nabla^2\Psi$. This yields

$$\nabla^2\Psi = \frac{\rho}{\epsilon} + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) \qquad \text{or, equivalently,} \qquad \nabla^2\Psi - \frac{\partial(\nabla \cdot \boldsymbol{A})}{\partial t} = \frac{\rho}{\epsilon}.$$

4. An electric field is given by $\boldsymbol{E}(t, x, y, z) = \sin(\omega t - kz)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ in the source-free case for which $\rho(t, x, y, z) = 0$, $\boldsymbol{J}(t, x, y, z) = \boldsymbol{0}$, for all (t, x, y, z), where ω is a constant. Let Γ be the circular boundary of a disc of radius a in the xy-plane with centre at the origin and counterclockwise around the z-axis. If $\boldsymbol{B}(t, x, y, z)$ is the corresponding magnetic field, evaluate the circulation of the magnetic field around Γ as a function of t. That is, evaluate the line integral

$$\oint_\Gamma \boldsymbol{B} \cdot d\boldsymbol{r}$$

as a function of t.

Solution. Let Σ denote the region of the disc of radius a on the xy-plane centered at the origin such that $\partial\Sigma = \Gamma$. Maxwell's equations state that

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}.$$

Since $\boldsymbol{J} = \boldsymbol{0}$ and $\frac{\partial \boldsymbol{E}}{\partial t} = \omega\cos\omega t(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$, we have that $\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\omega\cos\omega t(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$. By Stokes' Theorem, it follows that

$$\begin{aligned}
\oint_\Gamma \boldsymbol{B} \cdot \boldsymbol{r} &= \iint_\Sigma \nabla \times \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, dA \\
&= \mu_0\epsilon_0\omega\cos\omega t \iint_\Sigma (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}}\, dA \\
&= \mu_0\epsilon_0\omega\cos\omega t \iint_\Sigma 1\, dA \\
&= \pi a^2\mu_0\epsilon_0\omega\cos\omega t,
\end{aligned}$$

where we note that the normal vector to the disc is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} = (0, 0, 1)$ and the area of the disc of radius a is $\iint_\Sigma 1\, dA = \pi a^2$.

5. The most general form of a *plane wave* is given by

$$\boldsymbol{E}(t,\boldsymbol{r}) = f(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{E}_0$$

where $\boldsymbol{k} = (k_1, k_2, k_3)$ is the *wave vector* (which is *not* the same thing as $\hat{\boldsymbol{k}} = (0,0,1)$), f is an arbitrary function, $\boldsymbol{E}_0$ is a constant vector, and $\boldsymbol{r} = (x,y,z)$.

(a) Show that this field satisfies the wave equation $\dfrac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\nabla^2\boldsymbol{E} = \boldsymbol{0}$ as long as the dispersion relation $\omega^2 = c^2k^2$ is satisfied, where $k^2 = k_1^2 + k_2^2 + k_3^2 = \|\boldsymbol{k}\|^2$.

Solution. To keep the algebra clean, introduce a function $a(t,\boldsymbol{r}) = \omega t - \boldsymbol{k}\cdot\boldsymbol{r}$, which is called the *phase function*. Note that $\boldsymbol{k}\cdot\boldsymbol{r} = k_1x + k_2y + k_3z$. The useful times derivatives of the phase function are

$$\frac{\partial a}{\partial t} = \omega \quad \text{and} \quad \frac{\partial^2 a}{\partial t^2} = 0$$

while the useful space derivatives of the phase function are

$$\frac{\partial a}{\partial x} = -k_1, \quad \frac{\partial a}{\partial y} = -k_2, \quad \frac{\partial^2 a}{\partial z^2} = -k_3, \quad \text{and} \quad \frac{\partial^2 a}{\partial x^2} = \frac{\partial^2 a}{\partial y^2} = \frac{\partial^2 a}{\partial z^2} = 0.$$

We can write the electric field as $\boldsymbol{E}(t,\boldsymbol{r}) = f(a(t,\boldsymbol{r}))\boldsymbol{E}_0$. Note that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}f(a(t,\boldsymbol{r})) &= \frac{\partial}{\partial t}\left(f'(a(t,\boldsymbol{r}))\frac{\partial}{\partial t}a(t,\boldsymbol{r})\right)\\ &= f''(a(t,\boldsymbol{r}))\underbrace{\left(\frac{\partial a}{\partial t}\right)^2}_{=\omega^2} + f'(a(t,\boldsymbol{r}))\underbrace{\frac{\partial^2 a}{\partial t^2}}_{=0} = \omega^2 f''(a(t,\boldsymbol{r})),\end{aligned}$$

and thus

$$\begin{aligned}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} &= \left(\frac{\partial^2}{\partial t^2}f(a(t,\boldsymbol{r}))\right)\boldsymbol{E}_0\\ &= \omega^2 f''(a(t,\boldsymbol{r}))\boldsymbol{E}_0\end{aligned}$$

On the other hand, note that

$$\begin{aligned}\frac{\partial^2}{\partial x^2}f(a(t,\boldsymbol{r})) &= \frac{\partial}{\partial x}\left(f'(a(t,\boldsymbol{r}))\frac{\partial}{\partial x}a(t,\boldsymbol{r})\right)\\ &= f''(a(t,\boldsymbol{r}))\underbrace{\left(\frac{\partial a}{\partial x}\right)^2}_{=k_1^2} + f'(a(t,\boldsymbol{r}))\underbrace{\frac{\partial^2 a}{\partial x^2}}_{=0} = k_1^2 f''(a(t,\boldsymbol{r})),\end{aligned}$$

and similarly that

$$\frac{\partial^2}{\partial y^2}f(a(t,\boldsymbol{r})) = k_2^2 f''(a(t,\boldsymbol{r})) \qquad \text{and} \qquad \frac{\partial^2}{\partial z^2}f(a(t,\boldsymbol{r})) = k_3^2 f''(a(t,\boldsymbol{r})).$$

Hence

$$\begin{aligned}\nabla^2\boldsymbol{E} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\boldsymbol{E}\\ &= f''(a(t,\boldsymbol{r}))(k_1^2 + k_2^2 + k_3^2)\boldsymbol{E}_0\\ &= k^2 f''(a(t,\boldsymbol{r}))\boldsymbol{E}_0\end{aligned}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$. Thus,

$$\begin{aligned}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\nabla^2\boldsymbol{E} &= \omega^2 f''(a(t,\boldsymbol{r}))\boldsymbol{E}_0 - c^2k^2 f''(a(t,\boldsymbol{r}))\boldsymbol{E}_0 \\ &= \underbrace{(\omega^2 - c^2k^2)}_{=0} f''(a(t,\boldsymbol{r}))\boldsymbol{E}_0 \\ &= \boldsymbol{0},\end{aligned}$$

where we make use of the dispersion relation $\omega^2 = c^2k^2$.

(b) Although the wave equation is derived from Maxwell's equations, this does not imply that every solution of the wave equation is a solution of Maxwell's equations. In general, other conditions must be satisfied. What other condition must the plane wave above satisfy in order to be also a solution of Maxwell's equations?

Solution. Let $\boldsymbol{E}_0 = (E_{01}, E_{02}, E_{03})$. By the first Maxwell equation, we must have $\nabla\cdot\boldsymbol{E} = 0$. Let's check if the above wave satisfies this equation.

$$\begin{aligned}\nabla\cdot\boldsymbol{E} = \nabla\cdot(\boldsymbol{E}_0 f(a)) &= \frac{\partial}{\partial x}(E_{01}f(a)) + \frac{\partial}{\partial y}(E_{02}f(a)) + \frac{\partial}{\partial z}(E_{03}f(a)) \\ &= E_{01}f'(a)\frac{\partial a}{\partial x} + E_{02}f'(a)\frac{\partial a}{\partial y} + E_{03}f'(a)\frac{\partial a}{\partial z} \\ &= (E_{01}k_1 + E_{02}k_2 + E_{03}k_3)f'(a) \\ &= (\boldsymbol{E}_0\cdot\boldsymbol{k})f'(a)\end{aligned}$$

Since in general $f'(a) \neq 0$, the plane wave given will satisfy Maxwell's equations if and only if $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$. This means, the direction of the wave $\boldsymbol{k}$ is perpendicular to the direction in which $\boldsymbol{E}$ points.

(c) As a further example, show that the $\boldsymbol{E}$-field

$$\boldsymbol{E}(x,y,z,t) = (0, 0, E_0\cos(\omega t - kz))$$

satisfies the wave equation but not Maxwell's equations.

Solution. This field is a plane wave with wave vector $\boldsymbol{k} = \hat{\boldsymbol{k}} = (0,0,1)$ and $\boldsymbol{E_0} = (0,0,E_0)$, so it satisfies the wave equation. Comparing with part (b) above, in this case we have $\hat{\boldsymbol{k}}\cdot\boldsymbol{E}_0 = E_0 \neq 0$, unless $E_0 = 0$ in which case the field is the zero vector field. Hence this satisfies the wave equation but not Maxwell's equations, unless $E_0 = 0$.

6. A static electric field $\boldsymbol{E}(x,y,z)$ is caused by the charge distribution $\rho(x,y,z)$. The potential function is given by

$$\Psi(\boldsymbol{r}) = \begin{cases} -\dfrac{\rho_0 R^3}{3\epsilon_0 r}, & r \geq R \\ \dfrac{\rho_0}{6\epsilon_0}r^2 - \dfrac{\rho_0}{2\epsilon_0}R^2, & r < R \end{cases}$$

where $r = \sqrt{x^2+y^2+z^2}$.

(a) Find the electric field $\boldsymbol{E}(\boldsymbol{r})$.

Solution. Consider $r \geq R$. Then

$$\boldsymbol{E} = \nabla \Psi = \frac{\partial \Psi}{\partial r} \nabla r = \frac{\rho_0 R^3}{3\epsilon_0 r^2} \frac{\boldsymbol{r}}{r} = \frac{\rho_0 R^3}{3\epsilon_0 r^3} \boldsymbol{r}$$

If $r < R$, then

$$\boldsymbol{E} = \nabla \Psi = \frac{\partial \Psi}{\partial r} \nabla r = \frac{\rho_0 r}{3\epsilon_0} \frac{\boldsymbol{r}}{r} = \frac{\rho_0}{3\epsilon_0} \boldsymbol{r}$$

So the electric field is given by:

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} \dfrac{\rho_0 R^3}{3\epsilon_0 r^3} \boldsymbol{r}, & r \geq R \\ \dfrac{\rho_0}{3\epsilon_0} \boldsymbol{r}, & r < R \end{cases}$$

(b) Find the charge distribution $\rho(x, y, z)$.

Solution. From Maxwell's first equation, we know that $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$, and thus $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$. For $r \geq R$, note that $\boldsymbol{E}$ is the product of a scalar and vector field, so use the product rule for divergence to find that

$$\nabla \cdot \boldsymbol{E} = \nabla \left(\frac{\rho_0 R^3}{3\epsilon_0 r^3} \right) \cdot \boldsymbol{r} + \frac{\rho_0 R^3}{3\epsilon_0 r^3} (\nabla \cdot \boldsymbol{r}).$$

Now,

$$\nabla \left(\frac{\rho_0 R^3}{3\epsilon_0 r^3} \right) = \frac{d}{dr} \left(\frac{\rho_0 R^3}{3\epsilon_0 r^3} \right) \nabla r = \left(\frac{-3\rho_0 R^3}{3\epsilon_0 r^4} \right) \frac{\boldsymbol{r}}{r} = \frac{-\rho_0 R^3}{\epsilon_0 r^5} \boldsymbol{r}$$

Moreover, note that $\nabla \cdot \boldsymbol{r} = 3$. Substituting back in, we obtain

$$\nabla \cdot \boldsymbol{E} = \frac{-\rho_0 R^3}{\epsilon_0 r^5} \boldsymbol{r} \cdot \boldsymbol{r} + \frac{\rho_0 R^3}{\epsilon_0 r^3} = 0$$

since $\boldsymbol{r} \cdot \boldsymbol{r} = r^2$. For $r < R$, we have

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_0}{3\epsilon_0} \nabla \cdot \boldsymbol{r} = \frac{\rho_0}{\epsilon_0}.$$

Thus, the charge distribution function is given by

$$\rho(x, y, z) = \epsilon_0 \nabla \cdot \boldsymbol{E} = \begin{cases} 0, & r \geq R \\ \rho_0, & r < R. \end{cases}$$

Note this represents a uniformly distributed charge inside the sphere of radius R.

(c) Compute the surface integral $\iint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}} \, dA$ where Σ is the sphere $x^2 + y^2 + z^2 = 4R^2$ for $R > 0$ constant.

Solution. The surface is closed, so the Divergence Theorem applies:

$$\iint_\Sigma \boldsymbol{E} \cdot \hat{\boldsymbol{n}} \, dA = \iiint_\Omega \nabla \cdot \boldsymbol{E} \, dV$$

where Ω is the region inside the sphere of radius $2R$ such that $\partial\Omega = S$. Notice however that outside the sphere of radius R, the divergence is everywhere zero, meaning that

$$\iiint_\Omega \nabla \cdot \boldsymbol{E} \, dV = \iiint_{\Omega_R} \nabla \cdot \boldsymbol{E} \, dV$$

where Ω_R is the region inside the sphere of radius R.
Substituting in from (b),

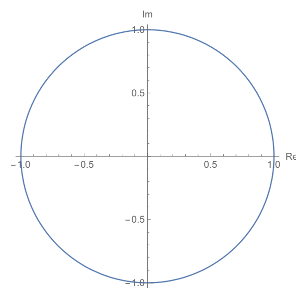
$$\iiint_{\Omega_R} \nabla \cdot \boldsymbol{E} \, dV = \iiint_{\Omega_R} \frac{\rho}{\epsilon_0} \, dV = \frac{\rho_0}{\epsilon_0} \iiint_{\Omega_R} dV = \frac{\rho_0}{\epsilon_0} \frac{4}{3} \pi R^3$$

2 Complex numbers

1. Sketch the set of points in the complex plane satisfying:

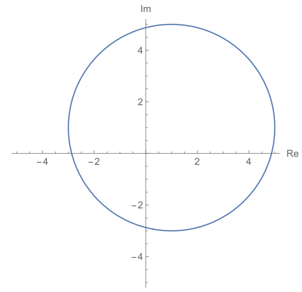
   (a) $|z| = 1$

   Solution. In Cartesian coordinates $z = x + jy$, this is $|z| = \sqrt{x^2 + y^2} = 1$, or equivalently $x^2 + y^2 = 1$. This is the unit circle in the complex plane that is centered at the origin.



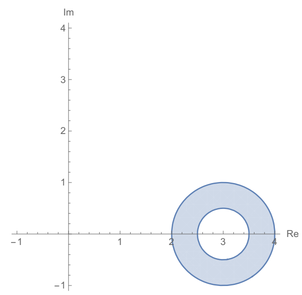
   (b) $|z - j - 1| = 4$

   Solution. We can write this as $|z - j - 1| = \sqrt{(x-1)^2 + (y-1)^2} = 2$, or equivalently $(x-1)^2 + (y-1)^2 = 4$. This is the circle in the complex plane of radius 2 that is centered at the point $1 + j$.



(c) $1 \leq |2z - 6| \leq 2$

Solution. We can rewrite this equation as $\frac{1}{2} \leq \sqrt{(x-3)^2 + y^2} \leq 1$. This defines the annulus centered at the point $z = 3$ with inner radius $\frac{1}{2}$ and outer radius 1.



(d) $|z - j|^2 + |z + j|^2 \leq 2$

Solution. In Cartesian coordinates with $z = x + jy$, the left-hand side can be written as

$$\begin{aligned} |z - j|^2 + |z + j|^2 &= x^2 + (y-1)^2 + x^2 + (y+1)^2 \\ &= x^2 + y^2 - 2y + 1 + x^2 + y^2 + 2y + 1 \\ &= 2(x^2 + y^2) + 2, \end{aligned}$$

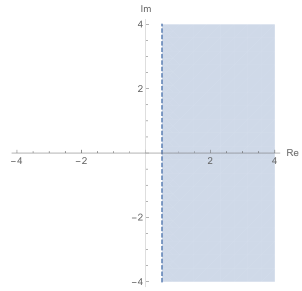
which is never less than 2. Moreover, it is equal to 2 if and only if $x = 0$ and $y = 0$. Hence the only point satisfying this equation is $z = 0$.

(e) $|z - 1| < |z|$

Solution. This is equivalent to $|z|^2 - |z-1|^2 > 0$. In Cartesian coordinates $z = x + jy$, this is

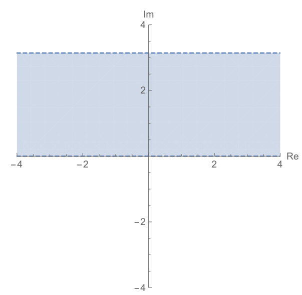
$$\begin{aligned} 0 < |z|^2 - |z-1|^2 &= x^2 + y^2 - \left((x-1)^2 + y^2\right) \\ &= 2x - 1, \end{aligned}$$

which is satisfied whenever $x > \frac{1}{2}$, i.e., whenever $\mathrm{Re}(z) > \frac{1}{2}$.



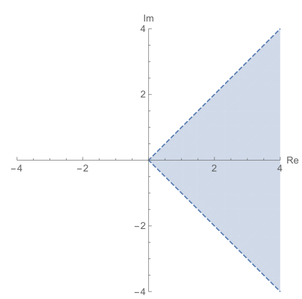
(f) $0 < \text{Im}(z) < \pi$

Solution. The complex numbers satisfying this inequality are simply the ones $z = x + jy$ with $0 < y < \pi$.



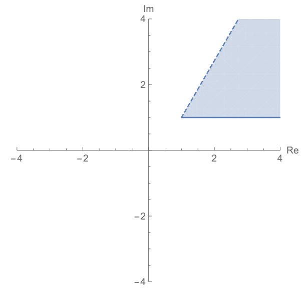
(g) $|\text{Arg}(z)| < \frac{\pi}{4}$

Solution. We use polar coordinates $z = re^{j\theta}$. The complex numbers satisfying this inequality are the ones with $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.



(h) $0 \leq \text{Arg}(z - j - 1) < \frac{\pi}{3}$

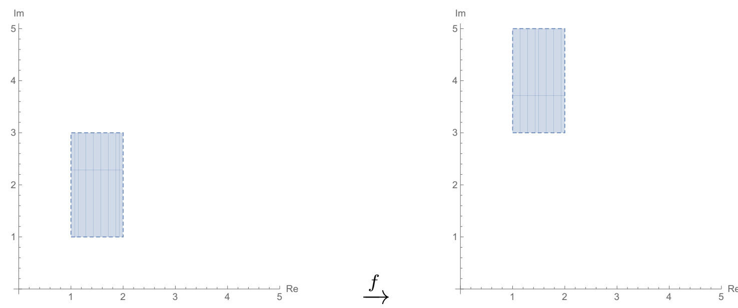
Solution. This is the set of complex numbers $re^{j\theta} + 1 + j$ with $0 \leq \theta < \frac{\pi}{3}$.



2. Let f be a mapping of the complex plane and let $A \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$ where f is defined. The *image* of A under f is the set of values $f(A) = \{f(z) : z \in A\}$. For each set A below, find and sketch the image $f(A)$ under the given mapping f.

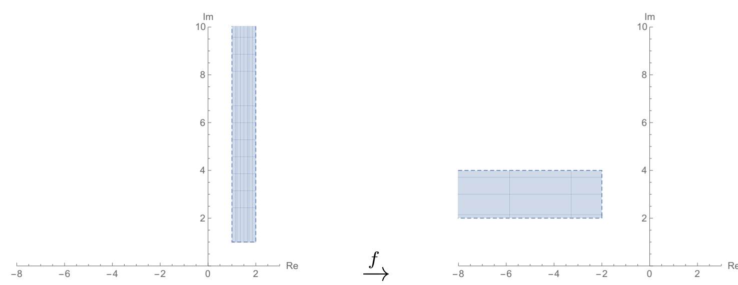
   (a) the set $A = \{x + jy : 1 < x < 2,\, 1 < y < 3\}$ under the mapping $f(z) = z + 2j$

   Solution. This mapping shifts everything up by 2.

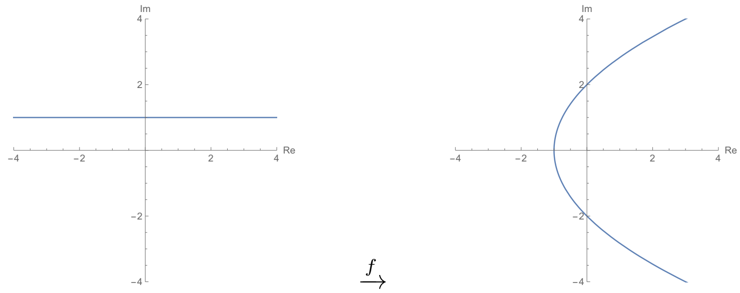
   (b) the set $A = \{x + jy : 1 < x < 2,\, 1 < y < \infty\}$ under the mapping $f(z) = 2jz$

   Solution. Note that we can write $2j = 2e^{j\pi/2}$. In polar coordinates $z = re^{j\theta}$, this mapping acts as $f(re^{j\theta}) = 2e^{j\pi/2}re^{j\theta} = 2re^{j(\theta+\pi/2)}$, which rotates z by $\frac{\pi}{2}$ and multiplies its modulus by 2. This mapping can be visualized as follows.

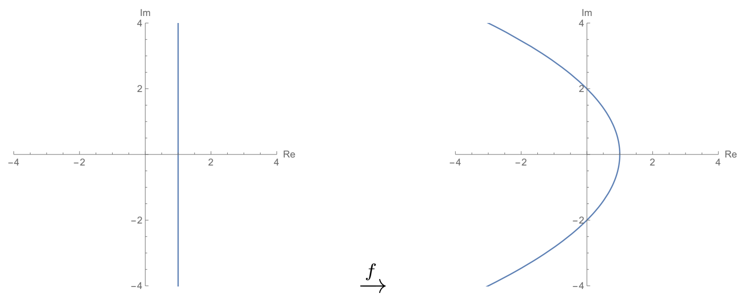
   (c) the set $A = \{z : \text{Im}(z) = 1\}$ under the mapping $f(z) = z^2$

   Solution. This maps the points of the form $z = x + j$ (i.e., with $y = 1$) in the z-plane to the set of points of the form $w = z^2 = (x + j)^2 = x^2 - 1 + j2x = u + jv$, where $u = x^2 - 1$ and $v = 2x$. This is the set of points of the form $u = \frac{1}{4}v^2 - 1$, which is a parabola in the w-plane.



(d) the set $A = \{z : \mathrm{Re}(z) = 1\}$ under the mapping $f(z) = z^2$

Solution. This maps the points of the form $z = 1 + jy$ (i.e., with $x = 1$) in the z-plane to the set of points of the form $w = z^2 = (1 + jy)^2 = 1 - y^2 + j2y = u + jv$, where $u = 1 - y^2$ and $v = 2y$. This is the set of points of the form $u = 1 - \frac{1}{4}v^2$, which is also a parabola in the w-plane, but opening in the other direction.



(e) the set $A = \{x + jy : 1 < x < 2,\ 1 < y < 3\}$ under the mapping $f(z) = z^2$

Solution. This region is bounded by the lines

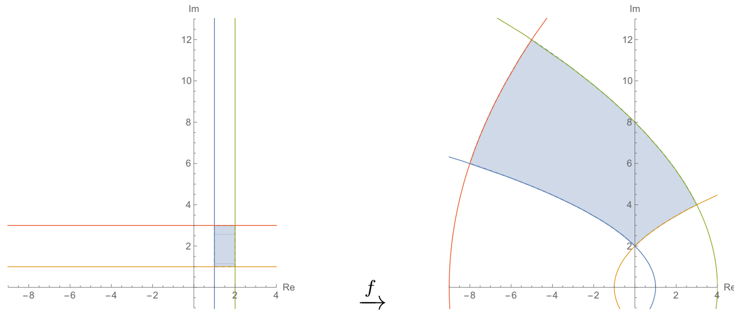
$$y = 1, \qquad y = 3, \qquad x = 1, \qquad \text{and} \qquad x = 2,$$

which in the z-plane we can write as $z = x + j$, $z = x + 3j$, $z = 1 + jy$, and $z = 2 + jy$. Each of these lines are mapped to different parabolas. From parts (c) and (d), we see that the lines $z = x + j$ and $z = 1 + jy$ are mapped to the parabolas $u = \frac{1}{4}v^2 - 1$ and $u = 1 - \frac{1}{4}v^2$. Similarly, the lines $z = x + 3j$ and $z = 2 + jy$ are mapped to

$$u + jv = (x + 3j)^2 = x^2 - 9 + j6x \qquad u + jv = (2 + jy)^2 = 4 - y^2 + j4y$$

which, in the w-plane, are the curves $u = \frac{1}{36}v^2 - 9$ and $u = 4 - \frac{1}{16}v^2$ respectively. Hence the image is the region bounded by the parabolas

$$u = \frac{1}{4}v^2 - 1, \qquad u = 1 - \frac{1}{4}v^2, \qquad u = \frac{1}{36}v^2 - 9, \qquad \text{and} \qquad u = 4 - \frac{1}{16}v^2.$$



3. Where are the following functions of a complex variable defined? (i.e. find their domains)

(a) $f(z) = \dfrac{z}{z + \bar{z}}$

Solution. Expanding z in Cartesian coordinates $z = x + jy$, we have

$$f(z) = \frac{x + jy}{(x + jy) + (x - jy)} = \frac{x + jy}{2x}.$$

Clearly, f is defined on all complex numbers with $\mathrm{Re}(z) \neq 0$. Geometrically, this is all of the complex plane except for the Im axis.

(b) $f(z) = \dfrac{1}{4 - |z|^2}$

Solution. We require that the denominator not be equal to zero. It holds that $4 - |z|^2 \neq 0$ if and only if $|z| \neq 2$. Geometrically, the domain is all of the complex plane except the circle centered at the origin of radius 2.