ECE 206 Fall 2019
Practice Problems Week 9
Solutions

1. Simplify the following expressions using properties of the complex exponential function.

   (a) $e^{2\pm 3\pi j}$

   Solution.

   $$e^{2\pm 3\pi j} = e^2 e^{\pm 3\pi j} = e^2(\cos(\pm 3\pi) + j\sin(\pm 3\pi)) = e^2(-1+0j) = -e^2$$

   (b) $e^{z+\pi j}$ for arbitrary $z \in \mathbb{C}$.

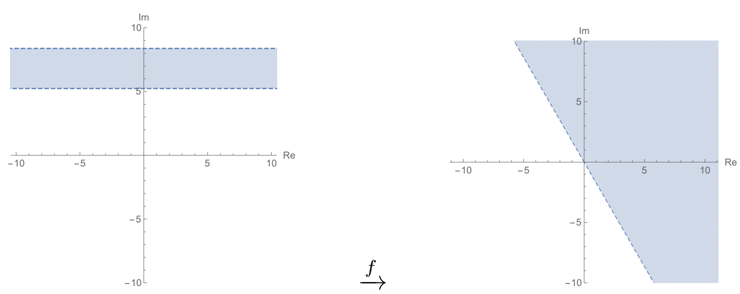
   Solution.

   $$e^{z+\pi j} = e^z e^{\pi j} = e^z(\cos(\pi) + j\sin(\pi)) = -e^z$$

2. Let f be a mapping of the complex plane and let $A \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$ where f is defined. The *image* of A under f is the set of values $\{f(z) \,|\, z \in A\}$. For each set below, sketch the set then find and sketch its image under the given mapping.

   (a) the set $A = \left\{ z \,\middle|\, \frac{5\pi}{3} < \mathrm{Im}(z) < \frac{8\pi}{3} \right\}$ under the mapping $f(z) = e^z$

   Solution. This maps all complex numbers of the form $z = x + jy$ with $\frac{5\pi}{3} < y < \frac{8\pi}{3}$ to complex numbers $e^z = e^x e^{jy}$. This is simply all complex numbers that have an argument in the range $(\frac{5\pi}{3}, \frac{8\pi}{3})$, or equivalently, with principal argument in the range $(-\frac{\pi}{3}, \frac{2\pi}{3})$.

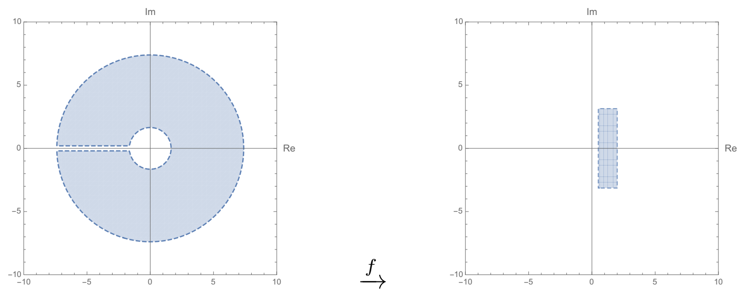
   

   (b) the slit annulus $A = \left\{ z \,\middle|\, \sqrt{e} \le |z| \le e^2 \text{ but } z \notin [-e^2, -\sqrt{e}) \right\}$ under $f(z) = \mathrm{Log}(z)$

   Solution. We can describe the slit annulus in polar coordinates as the set

   $$\{re^{j\theta} \,|\, \sqrt{e} \le r \le e^2,\ -\pi < \theta < \pi\}.$$

   The principal logarithm of these complex numbers is $\mathrm{Log}(re^{j\theta}) = \ln r + j\theta$, where the real and imaginary parts of this are in the ranges $\frac{1}{2} \le \ln r \le 2$ and $-\pi < \theta < \pi$.



3. Solve the following for all possible values of z.

(a) $e^z = -2$

Solution. Note that $-2 = 2e^{j\pi}$. Expanding $z = x_j y$ in Cartesian coordinates, we have $e^{x+jy} = -2 = 2e^{j\pi}$ or equivalently $e^x e^{jy} = 2e^{j\pi}$. Equating each part of the expression, we have $e^x = 2$ and thus $x = \ln 2$, while $e^{jy} = e^{j\pi}$ implies that $y = \pi + 2n\pi$, for $n = 0, \pm1, \pm2 \ldots$. Thus, the solution set is

$$z = \ln 2 + j\pi(1 + 2n), \qquad n = 0, \pm1, \pm2 \ldots$$

(b) $e^z = 1 + \sqrt{3}j$

Solution. We first must write the right-hand side in its polar form as

$$1 + \sqrt{3}j = 2e^{j\pi/3}.$$

This yields $e^x e^{jy} = 2e^{j\pi/3}$, which again leads to $x = \ln 2$ and $y = \frac{\pi}{3} + 2n\pi$, for $n = 0, \pm1, \pm2 \ldots$. The solution set is therefore

$$z = \ln 2 + j\pi\left(\frac{1}{3} + 2n\right), \qquad n = 0, \pm1, \pm2, \ldots$$

(c) $e^{2z-1} = 1$

Solution. Expanding z in Cartesian coordinates, we have

$$2z - 1 = 2(x + jy) - 1 = 2x - 1 + 2jy.$$

Moreover, we can write 1 as $1 = e^{2\pi j}$. Equating, we see that $e^{2x-1}e^{2jy} = e^{2\pi j}$ from which we find that $e^{2x-1} = 1$, hence $2x - 1 = 0$ and thus$x = \frac{1}{2}$. Furthermore, we must have that $2y = 2\pi + 2n\pi$ for some integer n, or $y = \pi(1 + n)$ for $n = 0, \pm1, \pm2, \ldots$.
Thus, the solution set is

$$z = \frac{1}{2} + j\pi(1 + n), \qquad n = 0, \pm1, \pm2, \ldots$$

(d) $\sin z = 3j$

Solution. In Cartesian coordinates $z = x + jy$, we expand $\sin z$ as

$$\begin{aligned}\sin z = \frac{e^{jz} - e^{-jz}}{2j} = \frac{e^{-y}e^{jx} - e^{y}e^{-jx}}{2j} &= \frac{e^{-y}(\cos x + j\sin x) - e^{y}(\cos x - j\sin x)}{2j} \\ &= \cos x\frac{e^{-y} - e^{y}}{2j} + j\sin x\frac{e^{y} + e^{-y}}{2j} \\ &= \sin x\cosh y + j\cos x\sinh y.\end{aligned}$$

So we need to solve $\sin x\cosh y + j\cos x\sinh y = 3j$ for x and y. Equating the real parts, we must have that $\sin x\cosh y = 0$. Note that $\cosh y$ is always positive. This means we must have $\sin x = 0$, or equivalently $x = m\pi$ for some integer m, and thus $\cos x = \pm 1$ (with sign depending on whether m is even or odd). Equating the imaginary parts, we have $3 = \pm\sinh y$, or

$$x = m\pi \text{ and } 3 = \sinh y \text{ if } m \text{ is even} \qquad \text{or} \qquad x = m\pi \text{ and } 3 = -\sinh y \text{ if } m \text{ is odd}.$$

We consider the two cases separately.

- We first consider the case when m is even and solve $3 = \sinh y$ for y. From the definition $\sinh y = \frac{e^{y}-e^{-y}}{2}$ and setting $\alpha = e^{y}$, we must solve $\alpha - \alpha^{-1} = 6$. Multiply both sides by α and rearrange to find that α must satisfy

  $$\alpha^2 - 6\alpha - 1 = 0 \qquad \text{or} \qquad \alpha = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \sqrt{10}.$$

  Now we have that $e^{y} = 3 \pm \sqrt{10}$. But $3 - \sqrt{10} < 0$ and e^{y} cannot be negative, so we must have $e^{y} = 3 + \sqrt{10}$ or equivalently $y = \ln(3 + \sqrt{10})$.

- Now consider the case when m is odd and solve $3 = -\sinh y$ for y. As before, we set $\alpha = e^{y}$ and solve $\frac{\alpha-\alpha^{-1}}{2} = -3$ for α. This is equivalent to solving

  $$\alpha^2 + 6\alpha - 1 = 0 \qquad \text{or} \qquad \alpha = \frac{-6 \pm \sqrt{36+4}}{2} = -3 \pm \sqrt{10}.$$

  Now we have that $e^{y} = -3 \pm \sqrt{10}$. But $-3 - \sqrt{10} < 0$ and e^{y} cannot be negative, so we must have $e^{y} = -3 + \sqrt{10}$ or equivalently $y = \ln(\sqrt{10} - 3)$.

Hence the solutions are of the form

$$z = m\pi + j\ln(\sqrt{10}+3) \text{ for even integers } m \qquad \text{and} \qquad z = m\pi + j\ln(\sqrt{10}-3) \text{ for odd integers } m.$$

(e) $\cos z = \cosh 4$

Solution. As above, we expand $\cos z$ in Cartesian coordinates $z = x + jy$ as

$$\begin{aligned}\cos z = \frac{e^{jz} + e^{-jz}}{2} = \frac{e^{-y}e^{jx} + e^{y}e^{-jx}}{2} &= \frac{e^{-y}(\cos x - j\sin x) + e^{y}(\cos x - j\sin x)}{2j} \\ &= \cos x\frac{e^{-y} + e^{y}}{2j} - j\sin x\frac{e^{y} - e^{-y}}{2j} \\ &= \cos x\cosh y - j\sin x\sinh y.\end{aligned}$$

Equating the imaginary parts, we find that x and y must satisfy $\sin x\sinh y = 0$, and thus either $\sin x = 0$ or $\sinh y = 0$. If $\sinh y = 0$ then $y = 0$ and the real parts must satisfy $\cosh 4 = \cos x$. But $\cosh 4 > 1$ and $\cos x \leq 1$ for all real values x, so we must have $\sin x = 0$. It follows that $x = m\pi$ for some integer m. Moreover, since $\cosh y$ is always positive, we must have $\cos x$ be positive, so x must be $x = m\pi$ for an *even* integer m (otherwise $\cos x = -1$). Finally, we need that $\cosh 4 = \cosh y$, so it must be that $y = \pm 4$. Hence all of the solutions are of the form

$$z = m\pi \pm 4j, \quad \text{where } m \text{ is an even integer.}$$

(f) $|\tan z| = 1$

Solution. Note that we can write $\tan z$ as

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{jz} - e^{-jz}}{2j} \frac{2}{e^{jz} + e^{-jz}} = \frac{1}{j} \frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}},$$

and thus

$$|\tan z| = \left| \frac{1}{j} \frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}} \right| = \frac{|e^{jz} - e^{-jz}|}{|e^{jz} + e^{-jz}|} = 1$$

is equivalent to $|e^{jz} - e^{-jz}| = |e^{jz} + e^{-jz}|$. We can square both sides to find that this is equivalent to

$$|e^{jz} - e^{-jz}|^2 = |e^{jz} + e^{-jz}|^2 \quad \text{or} \quad \left(e^{jz} - e^{-jz}\right) \overline{\left(e^{jz} - e^{-jz}\right)} = \left(e^{jz} + e^{-jz}\right) \overline{\left(e^{jz} + e^{-jz}\right)}.$$

Since $\overline{e^{jz} - e^{-jz}} = e^{-jz} - e^{jz}$ and $\overline{e^{jz} + e^{-jz}} = e^{-jz} + e^{jz}$, this is

$$\left(e^{jz} - e^{-jz}\right)\left(e^{-jz} - e^{jz}\right) = \left(e^{jz} + e^{-jz}\right)\left(e^{-jz} + e^{jz}\right)$$

and expanding yields

$$2 - e^{2jz} - e^{-2jz} = 2 + e^{2jz} + e^{-2jz}$$

or equivalently, after rearranging,

$$2\left(e^{2jz} + e^{-2jz}\right) = 0$$

which finally simplifies to $\cos(2z) = 0$. Solving this for z, we have

$$2z = \left(m + \frac{1}{2}\right)\pi \quad \text{for integers } m \in \mathbb{Z},$$

or equivalently $z = \frac{2m+1}{4}\pi$ for any integer $m \in \mathbb{Z}$.

(g) $\operatorname{Log} z = \frac{\pi}{2} j$

Solution. We use polar coordinates $z = re^{j\theta}$, with $-\pi < \theta \leq \pi$. Since this is the principal logarithm only, we have

$$\operatorname{Log} z = \operatorname{Log}(re^{j\theta}) = \ln r + j\theta = \frac{\pi}{2} j$$

which implies that $\ln r = 0$ and thus $r = 1$, and $\theta = \frac{\pi}{2}$. The only complex number that satisfies this is $z = e^{j\pi/2}$ or simply $z = j$.

4. Find all possible values of the following. Then find the principal value of each.

(a) $\log(-ej)$

Solution. Since $\mathrm{Arg}(-ej) = -\frac{\pi}{2}$, the argument of $-ej$ is the set

$$\arg(-ej) = \left\{ -\frac{\pi}{2} + 2n\pi \,\middle|\, n \in \mathbb{Z} \right\}.$$

Note that $|-ej| = e$ and thus $\ln|-ej| = 1$. Thus the logarithm is

$$\log(-ej) = \ln|-ej| + j\arg(-ej) = 1 + j\left(2n\pi - \frac{\pi}{2}\right) \qquad \text{for } n \in \mathbb{Z}.$$

The principal value of the logarithm is found by taking the principal argument,

$$\mathrm{Log}(-ej) = 1 - j\frac{\pi}{2}.$$

(b) $\log(1-j)$

Solution. Since the polar form of $1-j$ is $\sqrt{2}e^{j(-\pi/4)}$, we have

$$\log(1-j) = \ln\sqrt{2} + j\left(-\frac{\pi}{4} + 2n\pi\right) = \frac{1}{2}\ln 2 + j\pi\left(-\frac{1}{4} + 2n\right), \quad n \in \mathbb{Z}.$$

The principal logarithm is

$$\mathrm{Log}(1-j) = \frac{1}{2}\ln 2 - j\frac{\pi}{4}.$$

(c) $\log e$

Solution. We have

$$\log e = \ln|e| + j(\arg e + 2n\pi) = 1 + j\pi(0 + 2n) = 1 + 2n\pi j, \quad n \in \mathbb{Z},$$

and the principal logarithm is $\mathrm{Log}(e) = \ln e = 1$.

(d) $(-1)^{1/\pi}$

Solution. We use the properties of exponents to write this as $(-1)^{1/\pi} = e^{\frac{1}{\pi}\log(-1)}$. Since $-1 = e^{j\pi}$, we have that

$$\log(-1) = j\pi(2n+1) \quad \text{for } n \in \mathbb{Z}.$$

Thus, all of the possible values of this are

$$(-1)^{1/\pi} = e^{\frac{1}{\pi}\log(-1)} = e^{\frac{1}{\pi}(j\pi(2n+1))} = e^{j(2n+1)}$$

for $n \in \mathbb{Z}$. To find the principal value, note that $\mathrm{Log}(-1) = \pi$ and thus the principal value of $(-1)^{1/\pi}$ is $e^j = \cos 1 + j\sin 1$.

(e) $\left(\frac{e}{2}\left(-1 - \sqrt{3}j\right)\right)^{3\pi j}$

Solution. We can re-write this as

$$\left(\frac{e}{2}(-1-\sqrt{3}j)\right)^{3\pi j} = e^{3\pi j \log\left(\frac{e}{2}(-1-\sqrt{3}j)\right)}.$$

Note that $\left|\frac{e}{2}(-1-\sqrt{3}j)\right| = e$, while $\mathrm{Arg}(-1-\sqrt{3}j) = -\frac{2\pi}{3}$. Thus

$$\mathrm{Log}\left(\frac{e}{2}(-1-\sqrt{3}j)\right) = \ln\left|\frac{e}{2}(-1-\sqrt{3}j)\right| + j\,\mathrm{Arg}(-1-\sqrt{3}j) = \ln e - j\frac{2\pi}{3} = 1 - \frac{2\pi}{3}j.$$

Now, $3\pi j\,\mathrm{Log}\left(\frac{e}{2}(-1-\sqrt{3}j)\right) = 3\pi j\left(1-\frac{2\pi}{3}j\right) = 3\pi j + 2\pi^2$. Finally, the principal value is

$$\left(\frac{e}{2}(-1-\sqrt{3}j)\right)^{3\pi j} = e^{3\pi j\,\mathrm{Log}\left(\frac{e}{2}(-1-\sqrt{3}j)\right)} = e^{3\pi j + 2\pi^2} = e^{3\pi j}e^{2\pi^2} = -e^{2\pi^2}$$

5. In class, we derived the formula $\sin^{-1}(z) = -j\log\left(jz+\sqrt{1-z^2}\right)$ (where the equality is viewed as an equality of sets). Use similar methods to derive the following formulas.

(a) $\cos^{-1}(z) = -j\log\left(z+\sqrt{z^2-1}\right)$

Solution. We interpret $\cos^{-1}(z)$ as the set of values

$$\cos^{-1}(z) = \{w \in \mathbb{C} \mid \cos w = z\}.$$

To find all values w such that $\cos w = z$, expand $\cos w$ as

$$\cos w = \frac{e^{jw}+e^{-jw}}{2} = \frac{\alpha+\alpha^{-1}}{2}$$

where we define $\alpha = e^{jw}$. We first solve $\frac{\alpha+\alpha^{-1}}{2} = z$ for α. Multiplying both sides by α and rearranging, we see that this is equivalent to solving

$$\alpha^2 - 2z\alpha + 1 = 0, \qquad \text{and thus} \qquad \alpha = \frac{2z \pm \sqrt{4z^2-4}}{2} = z \pm \sqrt{z^2-1}.$$

Since $e^{jw} = \alpha = z \pm \sqrt{z^2-1}$, we have that w must be of the form $w = \frac{1}{j}\log(z \pm \sqrt{z^2-1})$, which is multi-valued.

(b) $\sinh^{-1}(z) = \log\left(z+\sqrt{1+z^2}\right)$

Solution. We interpret $\sinh^{-1}(z)$ as the set of values

$$\sinh^{-1}(z) = \{w \in \mathbb{C} \mid \sinh w = z\}.$$

To find all values w such that $\sinh w = z$, expand $\sinh w$ as

$$\sinh w = \frac{e^{w}-e^{w}}{2} = \frac{\alpha-\alpha^{-1}}{2}$$

where we define $\alpha = e^{w}$. We first solve $\frac{\alpha-\alpha^{-1}}{2} = z$ for α. Multiplying both sides by α and rearranging, we see that this is equivalent to solving

$$\alpha^2 - 2z\alpha - 1 = 0, \qquad \text{and thus} \qquad \alpha = \frac{2z \pm \sqrt{4z^2-4(-1)}}{2} = z \pm \sqrt{z^2+1}.$$

Since $e^w = \alpha = z \pm \sqrt{1+z^2}$, we have that w must be of the form $w = \log(z \pm \sqrt{1+z^2})$, which is multi-valued.

6. Determine where the following mappings are differentiable, and find the derivative $f'(z)$ at those values.

   (a) $f(z) = z - \bar{z}$

   Solution. Expanding $z = x + jy$ in Cartesian coordinates, we have,

   $$f(z) = f(x+jy) = x + jy - (x - jy) = 2jy.$$

   Thus, we have the real part $u(x,y) = 0$ and the imaginary part $v(x,y) = 2y$ of the function f such that $f(x+jy) = u(x,y) + jv(x,y)$. The partial derivatives of u and v are

   $$u_x(x,y) = u_y(x,y) = v_x(x,y) = 0 \qquad \text{and} \qquad v_y(x,y) = 2$$

   We see that the Cauchy-Riemann equations can never be satisfied, since

   $$u_x - v_y = -2 \neq 0 \text{ for all values of } x, y.$$

   Thus f is nowhere differentiable.

   (b) $f(z) = x^2 + jy^2$

   Solution. The real and imaginary parts of f are $u(x,y) = x^2$ and $v(x,y) = y^2$. The partial derivatives of u and v are

   $$u_y(x,y) = v_x(x,y) = 0, \qquad u_x(x,y) = 2x, \qquad \text{and} \quad v_y(x,y) = 2y.$$

   Checking the Cauchy-Riemann equations, we see that $u_y = -v_x$ is always satisfied, but

   $$u_x(x,y) = v_y(x,y) \quad \Longleftrightarrow \quad 2x = 2y \quad \Longleftrightarrow \quad y = x.$$

   Thus f is differentiable only on the line $y = x$. Since f is differentiable on this line, we may compute the derivative of f at points on this line as

   $$f'(x+jy) = u_x(x,y) + jv_x(x,y) = 2x,$$

   and thus $f'(z) = 2x$ along the line $y = x$.

   Remark. If a function $f : D \to \mathbb{C}$ is differentiable at some point $z_0 \in D$, the derivative $f'(z_0)$ at the point $z_0 = x_0 + jy_0$ may be computed as

   $$f'(z_0) = u_x(x_0, y_0) + jv_x(x_0, y_0).$$

   Since f is differentiable, the functions u and v must satisfy the Cauchy-Riemann equations, so the derivative can also be given by $f'(z_0) = v_y(x_0, y_0) - ju_y(x_0, y_0)$.

   (c) $f(z) = z\,\mathrm{Im}(z)$

Solution. Expanding in in Cartesian coordinates, the function can be written as $f(z) = f(x+jy) = (x+jy)(y) = xy+jy^2$. Hence the real and imaginary parts of f can be written as, $u(x,y) = xy$ and $v(x,y) = y^2$. The partial derivatives of u and v are

$$u_x = y, \qquad u_y = x, \qquad v_x = 0, \qquad \text{and} \qquad v_y = 2y.$$

It holds that $u_x - v_y = y$ and that $u_y + v_x = x$. Hence the Cauchy-Riemann equations hold if and only if $x = y = 0$. It follows that f is differentiable only at the origin $z = 0$. The derivative of f at this point is $f'(0) = u_x(0,0) + jv_x(0,0) = 0$.

7. Let $f : D \to \mathbb{C}$ be a complex-valued function on a domain $D \subseteq \mathbb{C}$. Show that if $f'(z) = 0$ everywhere in D, then f must be constant throughout D (i.e., there is some $\alpha \in \mathbb{C}$ such that $f(z) = \alpha$ for all $z \in D$).

Solution. Expand f in real and imaginary parts as $f(x+jy) = u(x,y)+jv(x,y)$. The derivative of f exists on all of D, so f is differentiable on D and its derivative may be given by both

$$f'(x+jy) = u_x(x,y) + jv_x(x,y) \qquad \text{and} \qquad f'(x+jy) = v_y(x,y) - ju_y(x,y)$$

(see the Remark in the solution to problem 3b above). By assumption, it holds that $f'(z) = 0$ for all $z \in D$ and thus $u_x = v_x = 0$ and $u_y = v_y = 0$. Stated another way, it holds that both $\nabla u = \mathbf{0}$ and $\nabla v = \mathbf{0}$ everywhere in D (i.e., their gradients are zero everywhere). Thus, both u and v must be constant $u(x,y) = a$ and $v(x,y) = b$, where $a, b \in \mathbb{R}$ are constants. Defining the constant $\alpha = a + jb$, we see that $f(z) = \alpha$ holds for all $z \in D$. This proves the claim.