ECE 206 Fall 2019
Practice Problems Week 10

1. (a) Using the transformation $x = r\cos\theta$, $y = r\sin\theta$ along with the chain rule, express the derivatives $u_r, u_\theta, v_r, v_\theta$ in terms of u_x, u_y, v_x, v_y to derive the polar form of the Cauchy-Riemann equations (CRE).

   (b) Find the formula for the derivative $f'(z) = e^{-j\theta}(u_r + jv_r)$ in polar coordinates. (Hint: start with the formula $f'(z) = u_x + jv_x$, and solve the equations from the previous part for u_x and v_x.)

   (c) Verify the CRE in polar form hold for the following mappings, and use the above to determine $f'(z)$.

      i. $f(z) = \dfrac{1}{z}$

      ii. $f(z) = \sqrt{z}$ (the principal value of the square root)

2. (a) Let $f(z) = u(r,\theta) + jv(r,\theta)$ be differentiable in a domain that does not include the origin. Starting from the Cauchy-Riemann equations in polar coordinates, show that the function $u(r,\theta)$ satisfies the partial differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

   which is the polar form of Laplace's equation. One can also show that the same equation holds for v. Assume that u, v are C^2.

   (b) Verify that $u(r,\theta) = \ln r$ is harmonic in the slit plane $D = \{re^{j\theta} \mid r > 0 \text{ and } -\pi < \theta < \pi\}$. (That is, show it satisfies the Laplace equation in polar form). Use this to show that the harmonic conjugate of u is $v(r,\theta) = \theta$.

3. For each of the following, a function u of variables x and y is given. Show that u can be the real part of some differentiable mapping $f(z)$. Determine the corresponding imaginary part v of f, and determine an expression of $f(z)$ purely in terms of z.

   (a) $u(x,y) = x^2 + 4x - y^2 + 2y$

   (b) $u(x,y) = \sinh x \sin y$

   (c) $u(x,y) = e^x \cos y - y$

4. Find the family of curves that is everywhere orthogonal to the family of curves $x^3y - xy^3 = c$ for constants $c \in \mathbb{R}$.

5. Find the equations for the families of level curves of the component functions u and v when $f(z) = \dfrac{1}{z}$. Make a sketch of a few level curves of each, indicating the orthogonality.

6. Use parametric representations for Γ to evaluate $\displaystyle\int_\Gamma \frac{z+2}{z}\,dz$, where Γ is

   (a) the semicircle $z = 2e^{j\theta}$ $\quad (0 \le \theta \le \pi)$

   (b) the semicircle $z = 2e^{j\theta}$ $\quad (\pi \le \theta \le 2\pi)$

   (c) the circle $z = 2e^{j\theta}$ $\quad (0 \le \theta \le 2\pi)$

7. Use parametric representations for Γ to evaluate $\displaystyle\int_\Gamma (z-1)\,dz$, where Γ is

(a) the semicircle $z = 1 + e^{j\theta} \quad (\pi \leq \theta \leq 2\pi)$

(b) the segment $0 \leq x \leq 2$ of the real axis

Why are the answers the same? Show another way to obtain the result.