ECE 206 Fall 2019
Practice Problems Week 10
Solutions

1. (a) Using the transformation $x = r\cos\theta$, $y = r\sin\theta$ along with the chain rule, express the derivatives $u_r, u_\theta, v_r, v_\theta$ in terms of u_x, u_y, v_x, v_y to derive the polar form of the Cauchy-Riemann equations (CRE).

Solution. By the chain rule,

$$u_r = u_x x_r + u_y y_r = \cos\theta u_x + \sin\theta u_y \tag{1}$$
$$u_\theta = u_x x_\theta + u_y y_\theta = -r\sin\theta u_x + r\cos\theta u_y \tag{2}$$
$$v_r = v_x x_r + v_y y_r = \cos\theta v_x + \sin\theta v_y \tag{3}$$
$$v_\theta = v_x x_\theta + v_y y_\theta = -r\sin\theta v_x + r\cos\theta v_y \tag{4}$$

Using the CRE, we can express (4) as

$$v_\theta = r(\sin\theta u_y + \cos\theta u_x) = r u_r$$

Similarly, we can write (2) as

$$u_\theta = -r(\sin\theta v_y + \cos\theta v_x) = -r v_r$$

This gives the CRE in polar form:

$$u_r = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = -v_r$$

(b) Find the formula for the derivative $f'(z) = e^{-j\theta}(u_r + jv_r)$ in polar coordinates. (Hint: start with the formula $f'(z) = u_x + jv_x$, and solve the equations from the previous part for u_x and v_x.)

Solution. We attempt to eliminate u_y from (1) and (2). This can be done by multiplying (1) through by $r\cos\theta$, multiplying (2) through by $\sin\theta$, and subtracting. This gives:

$$r\cos\theta u_r - \sin\theta u_\theta = r\cos^2\theta u_x + r\sin^2\theta u_x \quad \Longrightarrow \quad u_x = \cos\theta u_r - \frac{\sin\theta}{r} u_\theta$$

The same operations can be done to isolate for v_x:

$$v_x = \cos\theta v_r - \frac{\sin\theta}{r} v_\theta$$

Thus, $f'(z)$ can be expressed as

$$\begin{aligned}
f'(z) &= u_x + jv_x \\
&= \cos\theta u_r - \frac{\sin\theta}{r}u_\theta + j(\cos\theta v_r - \frac{\sin\theta}{r}v_\theta) \\
&= \cos\theta u_r + \sin\theta v_r + j(\cos\theta v_r - \sin\theta u_r) \qquad \text{using the CRE} \\
&= u_r(\cos\theta - j\sin\theta) + v_r(j\cos\theta + \sin\theta) \\
&= u_r e^{-j\theta} + jv_r(\cos\theta - \sin\theta) \\
&= e^{-j\theta}(u_r + jv_r)
\end{aligned}$$

(c) Verify the CRE in polar form hold for the following mappings, and use the above to determine $f'(z)$.

i. $f(z) = \dfrac{1}{z}$

Solution. With $z = re^{j\theta}$ in polar coordinates, we have

$$f(z) = \frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{r}\cos\theta - j\frac{1}{r}\sin\theta.$$

That is, $u(r,\theta) = \dfrac{\cos\theta}{r}$ and $v(r,\theta) = -\dfrac{\sin\theta}{r}$. Checking the CRE, we have

$$u_r = -\frac{\cos\theta}{r^2}, \quad u_\theta = -\frac{\sin\theta}{r}, \quad v_r = \frac{\sin\theta}{r^2}, \quad v_\theta = -\frac{\cos\theta}{r}$$

It is observed that the CRE are satisfied. Now, we use the formula for the derivative to find

$$\begin{aligned}
f'(z) = e^{-j\theta}(u_r + jv_r) &= e^{-j\theta}(-\frac{\cos\theta}{r^2} + j\frac{\sin\theta}{r^2}) \\
&= -\frac{e^{-j\theta}}{r^2}(\cos\theta - j\sin\theta) \\
&= -\frac{e^{-2j\theta}}{r^2} \\
&= -\frac{1}{(re^{j\theta})^2} \\
&= -\frac{1}{z^2}
\end{aligned}$$

ii. $f(z) = \sqrt{z}$ (the principal value of the square root)

Solution. Note here that $f(z) = \sqrt{z}$ is only continuous and differentiable on the *slit plane* $\mathbb{C} \setminus (-\infty, 0] = \{re^{j\theta} \,|\, r > 0 \text{ and } \theta \in (-\pi, \pi)\}$. For $r > 0$ and $-\pi < \theta < \pi$, we have

$$f(z) = \sqrt{r^{j\theta}}\sqrt{r}e^{j\theta/2} = \sqrt{r}\cos\frac{\theta}{2} + j\sqrt{r}\sin\frac{\theta}{2}.$$

That is, $u(r,\theta) = \sqrt{r}\cos\theta/2$ and $v(r,\theta) = \sqrt{r}\sin\theta/2$. The partial derivatives of u and

v are

$$u_r = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}, \quad u_\theta = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2}, \quad v_r = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad v_\theta = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2}.$$

It is observed that the CRE are satisfied. Now, we use the formula for the derivative to find

$$\begin{aligned} f'(z) = e^{-j\theta}(u_r + jv_r) &= e^{-j\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + j\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right) \\ &= \frac{1}{2\sqrt{r}}e^{-j\theta}e^{j\theta/2} \\ &= \frac{1}{2\sqrt{r}e^{j\theta/2}} = \frac{1}{2\sqrt{z}}. \end{aligned}$$

2. (a) Let $f(z) = u(r,\theta) + jv(r,\theta)$ be differentiable in a domain that does not include the origin. Starting from the Cauchy-Riemann equations in polar coordinates, show that the function $u(r,\theta)$ satisfies the partial differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

which is the polar form of Laplace's equation. One can also show that the same equation holds for v. Assume that u, v are C^2.

Solution. In polar form, the CRE are:

$$u_r = \frac{1}{r}v_\theta, \quad \frac{1}{r}u_\theta = -v_r$$

Differentiating the first equation with respect to r (using the product rule on the right):

$$u_{rr} = -\frac{1}{r^2}v_\theta + \frac{1}{r}v_{\theta r}$$

Differentiating the second equation with respect to θ:

$$\frac{1}{r}u_{\theta\theta} = -v_{r\theta}$$

Now, since $v \in C^2$, $v_{r\theta} = v_{\theta r}$. Also, we can use the first CRE for v_θ to get:

$$u_{rr} = -\frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} \Longrightarrow u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

(b) Verify that $u(r,\theta) = \ln r$ is harmonic in the slit plane $D = \{re^{j\theta} \,|\, r > 0 \text{ and } -\pi < \theta < \pi\}$. (That is, show it satisfies the Laplace equation in polar form). Use this to show that the harmonic conjugate of u is $v(r,\theta) = \theta$.

Solution. We have $u_r = \frac{1}{r}$, $u_{rr} = -\frac{1}{r^2}$, $u_\theta = u_{\theta\theta} = 0$. The Laplace equation gives:

$$-\frac{1}{r^2} + \frac{1}{r}\left(\frac{1}{r}\right) + 0 = 0$$

The harmonic conjugate must satisfy the CRE in polar coordinates. So, $u_r = \frac{1}{r}v_\theta \Longrightarrow$

$v_\theta = 1 \Longrightarrow v = \theta + g(r)$, where g is an arbitrary function. Now differentiate with respect to r: $v_r = g'(r)$, which from the second CRE must equal $\frac{1}{r}u_\theta$, which is 0. Thus, $g(r) = c$ for some constant c, which we may take to be zero. A harmonic conjugate is $v(r,\theta) = \theta$.

3. For each of the following, a function u of variables x and y is given. Show that u can be the real part of some differentiable mapping $f(z)$. Determine the corresponding imaginary part v of f, and determine an expression of $f(z)$ purely in terms of z.

(a) $u(x,y) = x^2 + 4x - y^2 + 2y$

Solution. We first check that u is harmonic. Note that $u_{xx} = 2$ and $u_{yy} = -2$, so $u_{xx} + u_{yy} = 0$ and thus u is indeed harmonic. In order to find a function v such that $f = u + jv$ is a differentiable mapping, the functions u and v must satisfy the Cauchy-Riemann equations. That is, u and v must satisfy

$$u_x = 2x + 4 = v_y \qquad \text{and} \qquad u_y = -2y + 2 = -v_x$$

It follows that v must be of the form

$$v(x,y) = \int (2x+4)\,dy = 2xy + 4y + g(x)$$

where g is a function of x. Differentiating with respect to x and comparing with the second Cauchy-Riemann equation above, it follows that

$$v_x(x,y) = 2y + g'(x) = -u_y = 2y - 2,$$

and thus $g'(x) = -2$. Hence $g(x) = -2x + c$ for some constant $c \in \mathbb{R}$ which we may take to be equal to 0. Thus $v(x,y) = 2xy + 4y - 2x$.

The corresponding complex differentiable is $f(x+jy) = x^2+4x-y^2+2y+j(2xy+4y-2x)$. Rearranging, we can write this as

$$\begin{aligned} f(z) = f(x+jy) &= \big(4x + 4jy\big) + \big(-j2x + 2y\big) + \big(x^2 - y^2 + j2xy\big) \\ &= 4(x+jy) - j2(x+jy) + (x+jy)^2 \\ &= 4z - j2z + z^2, \end{aligned}$$

and thus the differentiable mapping is $f(z) = 4z - j2z + z^2$ which has real part given by u.

Remark. Given a harmonic function u, a function v such that $u + jv$ is a differentiable mapping is called the *harmonic conjugate* of u.

In this problem, the function defined by $v(x,y) = 2xy + 4y - 2x$ is the harmonic conjugate of the function defined by $u(x,y) = x^2 + 4x - y^2 + 2y$.

(b) $u(x,y) = \sinh x \sin y$

Solution. We first check that u is harmonic. Note that

$$\frac{\partial^2}{\partial x^2} \sinh x = \sinh x \qquad \text{and} \qquad \frac{\partial^2}{\partial y^2} \sin y = -\sin y,$$

and thus $u_{xx} = \sinh x \sin y = u$ and $u_{yy} = -\sinh x \sin y - u$ so that u is indeed harmonic. We now find the function v that is the harmonic conjugate of u. By the Cauchy-Riemann equations, the function v must satisfy

$$u_x = \cosh x \sin y = v_y \qquad \text{and} \qquad u_y = \sinh x \cos y = -v_x$$

Integrating the first equation with respect to y, we get

$$v = \int \big(\cosh x \sin y\big)\, dy = -\cosh x \cos y + g(x)$$

where g is some function of x. Differentiating with respect to x y

$$v_x = -\sinh x \cos y + g'(x).$$

Comparing with the second Cauchy-Riemann equation above, we find that $g'(x) = 0$ or $g(x) = c$ for some constant $c \in \mathbb{R}$, which we may take to be zero. The harmonic conjugate of u is thus $v(x,y) = -\cosh x \cos y$, and the mapping defined by

$$f(z) = f(x + jy) = \sinh x \sin y - j(\cosh x \cos y)$$

is differentiable. It can be verified that $f(z)$ can be written as $f(z) = -j \cosh z$.

(c) $u(x,y) = e^x \cos y - y$

Solution. To see that u is harmonic, note that $u_{xx} = e^x \cos y$ and $u_{yy} = -e^x \cos y$, and so u is indeed harmonic. By the Cauchy-Riemann equations, the harmonic conjugate v must satisfy

$$u_x = e^x \cos y = v_y \qquad \text{and} \qquad u_y = -e^x \sin y - 1 = -v_x$$

Integrating the first equation with respect to y, we get

$$v = \int \big(e^x \cos y\big)\, dy = e^x \sin y + g(x)$$

where g is some function of x. Differentiating with respect to x gives

$$v_x = e^x \sin y + g(x).$$

Comparing with the second Cauchy-Riemann equation above, we find that $g'(x) = 1$ or $g(x) = x + c$ for some constant $c \in \mathbb{R}$, which we may take to be zero. The harmonic conjugate of u is thus $v(x,y) = e^x \sin y + x$, and the mapping defined by

$$\begin{aligned} f(z) = f(x + jy) &= e^x \cos y - y + j\left(e^x \sin y + x\right) \\ &= e^x(\cos y + j \sin x) - y + jx \\ &= e^{x+jy} + j(x + jy) \\ &= e^z + jz \end{aligned}$$

is differentiable.

4. Find the family of curves that is everywhere orthogonal to the family of curves $x^3y - xy^3 = c$ for constants $c \in \mathbb{R}$.

Solution. Recall from the lecture that if two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ are *harmonic conjugates*[a] of each other, then their level curves are orthogonal to each other. Define the function $u(x, y) = x^3y - xy^3$. We need to find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that u and v satisfy the Cauchy-Riemann equations (CRE). Taking the derivative of u with respect to x, we have $u_x = 3x^2 - y^3$. From the CRE, we need that $v_y = u_x$, and thus

$$v = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + g(x),$$

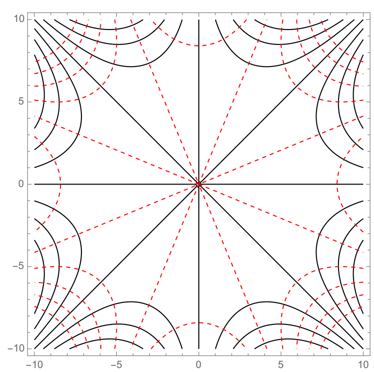
where g is an arbitrary function. We also require that $v_x = -u_y$, where the derivative of u with respect to y is $u_y = x^3 - 3xy^2$. Differentiating v with respect to x, we find $v_x = 3xy^2 + g'(x)$, and equating with $-u_y$ yields that $g'(x) = -x^3$, or $g(x) = -\frac{1}{4}x^4 + b$ for some constant $b \in \mathbb{R}$. Thus

$$v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}(x^4 + y^4) + b$$

Hence the family of curves that is orthogonal to the curves $x^3y - xy^3 = c$ can be given by $v(x, y) = a$, where $a \in \mathbb{R}$ is a constant, or equivalently (after multiplying everything by -4)

$$x^4 + y^4 - 6x^2y^2 = d$$

where $d = -4a$ is another constant. Some level curves of u (in black) and v (in red) are shown in the following figure.



(We may note that u and v are the real and imaginary components of the differentiable $f(z) = -j\frac{z^4}{4}$.)

[a] Two harmonic functions u and v are harmonic conjugates if they satisfy the Cauchy-Riemann equations.

5. Find the equations for the families of level curves of the component functions u and v when $f(z) = \dfrac{1}{z}$. Make a sketch of a few level curves of each, indicating the orthogonality.

Solution. Writing $f(z) = u(x, y) + jv(x, y)$ with $z = x + jy$, we have

$$\frac{1}{z} = \frac{1}{x + jy} = \frac{x - jy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - j\frac{y}{x^2 + y^2}.$$

That is, the components of f are

$$u(x,y) = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v(x,y) = -\frac{y}{x^2+y^2}.$$

The level curves $u = c_1$ for constants $c_1 \in \mathbb{R}$ are determined by

$$\frac{x}{x^2+y^2} = c_1 \quad \longrightarrow \quad x^2+y^2 = \frac{1}{c_1}x \quad \longrightarrow \quad \left(x - \frac{1}{2c_1}\right)^2 + y^2 = \frac{1}{4c_1^2}.$$

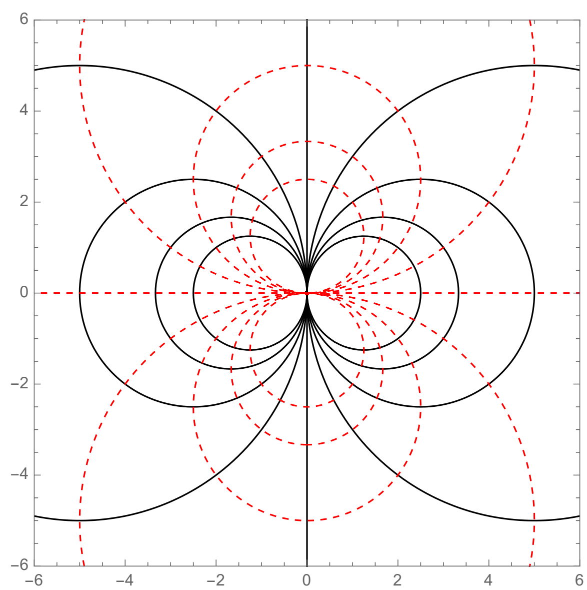
which is the equation for a circle centred at $(\frac{1}{2c_1}, 0)$ with radius $\frac{1}{2c_1}$, i.e. circles centred on the x-axis that pass through the origin.

The level curves $v = c_2$ for constants $c_2 \in \mathbb{R}$ are determined by

$$-\frac{y}{x^2+y^2} = c_2 \quad \longrightarrow \quad x^2+y^2 = -\frac{1}{c_2}y \quad \longrightarrow \quad x^2 + \left(y + \frac{1}{2c_2}\right)^2 = \frac{1}{4c_2^2}$$

which are circles centred at $(0, -\frac{1}{2c_2})$ with radius $\frac{1}{2c_2}$, i.e. circles centred on the $y-$axis that pass through the origin.

See a diagram with some of the level curves of u and v plotted below. Notice in the figure that all of the intersections are orthogonal.



Level curves of $u(x,y) = \frac{x}{x^2+y^2}$ (in black) and $v(x,y) = \frac{-y}{x^2+y^2}$ (in dashed red).

6. Use parametric representations for Γ to evaluate $\int_\Gamma \frac{z+2}{z}\,dz$, where Γ is

   (a) the semicircle $z = 2e^{j\theta} \quad (0 \le \theta \le \pi)$

Solution. Here we use the parameterization $\gamma(\theta) = 2e^{j\theta}$ which has derivative $\gamma'(\theta) = 2je^{j\theta}$. We may evaluate the desired integral as

$$\int_0^\pi \frac{(2+2e^{j\theta})}{2e^{j\theta}} 2je^{j\theta}\, d\theta = 2j \int_0^\pi \left(e^{j\theta}+1\right)\, d\theta = 2j\left(\frac{1}{j}(e^{j\pi}-1)+\pi\right) = -4+2\pi j$$

where $e^{j\pi} = \cos\pi = -1$.

(b) the semicircle $z = 2e^{j\theta} \quad (\pi \le \theta \le 2\pi)$

Solution. Same calculation as part (a), just different limits of integration.

$$\int_\pi^{2\pi} \frac{(2+2e^{j\theta})}{2e^{j\theta}} 2je^{j\theta}\, d\theta = \cdots = 2j\left(\frac{1}{j}(e^{2j\pi}-e^{j\pi})+\pi\right) = 4+2\pi j$$

(c) the circle $z = 2e^{j\theta} \quad (0 \le \theta \le 2\pi)$

Solution. The circle is made up of the two semi-circles from the previous parts, so this integral is simply the sum, which is $4\pi j$.

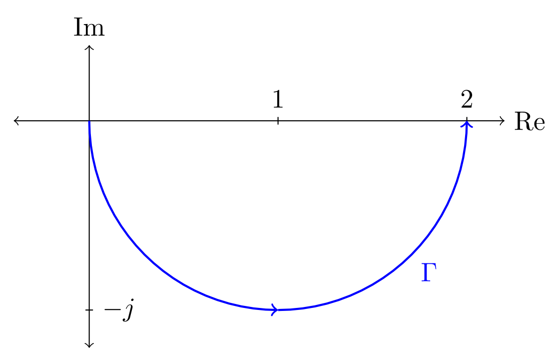
7. Use parametric representations for Γ to evaluate $\int_\Gamma (z-1)\, dz$, where Γ is

(a) the semicircle $z = 1 + e^{j\theta} \quad (\pi \le \theta \le 2\pi)$

Solution. We use the parameterization $\gamma(\theta) = 1 + e^{j\theta}$ for $\theta \in [\pi, 2\pi]$, which has derivative $\gamma'(\theta) = je^{j\theta}$. We therefore have

$$\int_\Gamma (z-1)\, dz = \int_\pi^{2\pi} \left(e^{j\theta}\right) je^{j\theta}\, d\theta = j\frac{1}{2j}e^{2j\theta}\Big|_\pi^{2\pi} = \frac{1}{2}(e^{4\pi j} - e^{2\pi j}) = 0.$$

A depiction of the path is shown below.



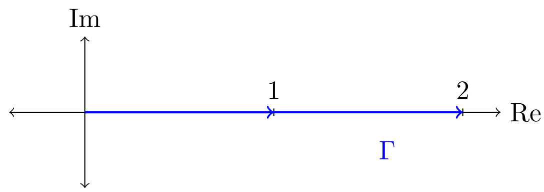
(b) the segment $0 \le x \le 2$ of the real axis

Solution. Here we use the parameterization $\gamma(t) = t$ for $t \in [0, 2]$, which has derivative

$\gamma'(t) = 1$. We therefore have

$$\int_\Gamma (z-1)\,dz = \int_0^2 (t-1)\,dt = \left[\frac{t^2}{2} - t\right]_{t=0}^2 = 0.$$

A depiction of the path is shown below.



Why are the answers the same? Show another way to obtain the result.

Solution. The results are the same because the endpoints of the paths are the same, and $f(z) = z - 1$ is a mapping that is differentiable everywhere.

We could have obtained this result using an anti-derivative of f. That is, since it holds that $\frac{d}{dz}\left(\frac{z^2}{2} - z\right) = z - 1$, we have

$$\int_\Gamma (z-1)\,dz = \int_0^2 (z-1)\,dz = \left[\frac{z^2}{2} - z\right]_0^2 = 0$$

for any path Γ whose initial and end points are 0 and 2 respectively.