ECE 206 Fall 2019
Practice Problems Week 11
Solutions

1. Let Γ be the contour defined by the path $\gamma(\theta) = e^{j\theta}$ for $-\pi \le \theta \le \pi$. Evaluate the following integrals.

   (a) $\displaystyle\int_\Gamma \operatorname{Log} z \, dz$

   Solution. The principal logarithm is differentiable on the "slit plane"

   $$D = \mathbb{C} \setminus (-\infty, 0] = \{re^{j\theta} \in \mathbb{C} \,|\, r > 0 \text{ and } -\pi < \theta < \pi\}$$

   which is $\mathbb{C}$ with the negative real axis removed. On the path $\gamma(\theta) = e^{j\theta}$ for θ in the open interval $\theta \in (-\pi, \pi)$, we have $\operatorname{Log}\gamma(\theta) = \operatorname{Log} e^{j\theta} = \ln|1| + j\theta = j\theta$. Also, we have $\gamma'(\theta) = je^{j\theta}$. Thus, the integral becomes

   $$\int_\Gamma \operatorname{Log} z \, dz = \int_{-\pi}^{\pi} (j\theta) je^{j\theta} \, d\theta = -\int_{-\pi}^{\pi} \theta e^{j\theta} \, d\theta.$$

   Here we use integration by parts with $u = \theta$, $dv = e^{j\theta} \, d\theta$, which leads to

   $$\begin{aligned}\int_\Gamma \operatorname{Log} z \, dz &= -\left(\frac{\theta}{j} e^{j\theta} \Big|_{-\pi}^{\pi} - \frac{1}{j} \int_{-\pi}^{\pi} e^{j\theta} \, d\theta \right) \\ &= j\left(-\pi - (-\pi(-1)) + j\frac{1}{j} e^{j\theta} \Big|_{-\pi}^{\pi} \right) \\ &= -2\pi j,\end{aligned}$$

   where $\frac{1}{j} = -j$ has been used, along with $e^{j\pi} = e^{-j\pi} = -1$.

   (b) $\displaystyle\int_\Gamma z^3 \operatorname{Log} z \, dz$

   Solution. Similar to above, we have

   $$\int_\Gamma z^3 \operatorname{Log} z \, dz = \int_{-\pi}^{\pi} e^{3j\theta} (j\theta) je^{j\theta} \, d\theta = -\int_{-\pi}^{\pi} \theta e^{4j\theta} \, d\theta.$$

   Again integrating by parts,

   $$\begin{aligned}\int_\Gamma \operatorname{Log} z \, dz &= -\left(\frac{\theta}{4j} e^{4j\theta} \Big|_{-\pi}^{\pi} - \frac{1}{4j} \int_{-\pi}^{\pi} e^{j\theta} \, d\theta \right) \\ &= \cdots = -\frac{\pi}{2} j.\end{aligned}$$

2. Let Γ denote the circle $|z - z_0| = R$, taken counterclockwise. Compute the following integrals using the path $\gamma(\theta) = z_0 + Re^{j\theta}$ for $\theta \in (-\pi, \pi)$.

(a) $\displaystyle\int_{\Gamma_0} \frac{1}{z - z_0}\, dz$

> *Solution.* The path $\gamma(\theta) = z_0 + Re^{j\theta}$ has derivative $\gamma'(\theta) = jRe^{j\theta}$. The integral becomes
>
> $$\int_\Gamma \frac{1}{z - z_0}\, dz = \int_{-\pi}^{\pi} \frac{jRe^{j\theta}}{Re^{j\theta}}\, d\theta = \int_{-\pi}^{\pi} j\, d\theta = 2\pi j.$$

(b) $\displaystyle\int_{\Gamma_0} (z - z_0)^{n-1}\, dz$, where $n \in \mathbb{Z}, n \neq 0$

> *Solution.* Similar to above, we have
>
> $$\int_\Gamma (z - z_0)^{n-1}\, dz = \int_{-\pi}^{\pi} (R^{n-1}e^{j(n-1)\theta})jRe^{j\theta}\, d\theta = jR^n \int_{-\pi}^{\pi} e^{jn\theta}\, d\theta = \frac{R^n}{n}(e^{jn\pi} - e^{-jn\pi}) = 0,$$
>
> since $e^{jn\pi} - e^{-jn\pi} = \cos n\pi - \cos(-n\pi) = 0$ for any integer $n \in \mathbb{Z}$.

(c) $\displaystyle\int_{\Gamma_0} (z - z_0)^{a-1}\, dz$ where $a \in \mathbb{R}$ is a constant with $a \neq 0$. Here, we take $(z - z_0)^{a-1}$ to be the principal value.

> *Solution.* This is essentially the same as the integral in (b), but just in the last step we no longer have that a is necessarily an integer. That is,
>
> $$\int_\Gamma (z - z_0)^{a-1}\, dz = \frac{R^a}{a}\left(e^{ja\pi} - e^{-ja\pi}\right) = \frac{R^a}{a}\big(\cos a\pi + j\sin a\pi - (\cos(a\pi) - j\sin(a\pi)\big)$$
> $$= j\frac{2R^a}{a}\sin a\pi.$$

3. Use anti-derivatives to evaluate the following integrals.

(a) $\displaystyle\int_j^{j/2} e^{\pi z}\, dz$

> *Solution.* Note that the function $f(z) = e^{\pi z}$ is differentiable everywhere and has anti-derivative given by $\frac{1}{\pi}e^{\pi z}$. Thus
>
> $$\int_j^{j/2} e^{\pi z}\, dz = \frac{1}{\pi}e^{\pi z}\Big|_j^{j/2} = \frac{1}{\pi}(e^{j\pi/2} - e^{j\pi}) = \frac{1}{\pi}(j - (-1)) = \frac{1+j}{\pi}.$$

(b) $\displaystyle\int_0^{\pi+2j} \cos\left(\frac{z}{2}\right)\, dz$

Solution. Note that $\cos\left(\frac{z}{2}\right)$ is differentiable everywhere with anti-derivative $-2\sin\left(\frac{z}{2}\right)$. Thus

$$\int_0^{\pi+2j} \cos\left(\frac{z}{2}\right)\,dz = 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2j} = 2\sin\left(\frac{\pi}{2}+j\right) = 2\cos j = 2\cosh 1 = e + \frac{1}{e},$$

where some identities have been used.

(c) $\displaystyle\int_j^{3j} (z-2j)^3\,dz$

Solution. Note that $(z-2j)^3$ is differentiable everywhere with anti-derivative $\frac{1}{4}(z-2j)^4$. Thus

$$\int_j^{3j} (z-2j)^3\,dz = \frac{1}{4}(z-2j)^4\Big|_j^{3j} = \frac{1}{4}(j^4 - (-j)^4) = 0.$$

4. Let Γ be the circle $|z| = 1$. For which of the following functions is $\displaystyle\int_\Gamma f(z)\,dz = 0$?

(a) $f(z) = z^3$

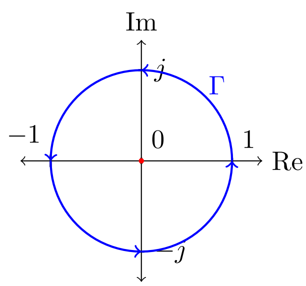
Solution. Here we see that $f(z) = z^3$ is differentiable on the entire complex plane. Thus $\displaystyle\oint_\Gamma f(z)\,dz = 0$ by the Cauchy-Goursat theorem.

(b) $f(z) = \dfrac{e^z}{z}$

Solution. Note that $f(z)$ is not differentiable at $z = 0$, which lies within Γ. Thus we may not use the Cauchy-Goursat theorem. However, we can define the function $g(z) = e^z$ (which is differentiable everywhere) and conclude from Cauchy's integral formula that

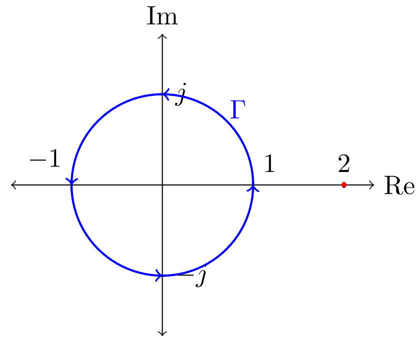
$$\oint_\Gamma \frac{e^z}{z}\,dz = \oint_\Gamma \frac{g(z)}{z-0}\,dz = 2\pi j g(0) = 2\pi j e^0 = 2\pi j,$$

which is not zero.



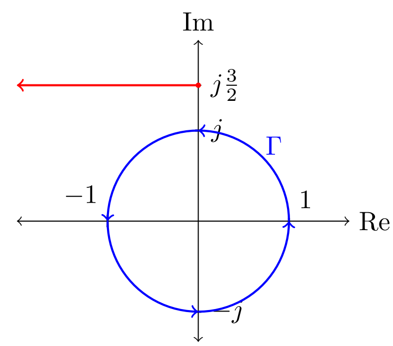
(c) $f(z) = \dfrac{e^z}{z-2}$

Solution. Note that f is differentiable everywhere except at the point $z = 2$. However, this point this lies outside of the region inside the curve, and thus $\int_\Gamma f(z)\,dz = 0$.



(d) $f(z) = \mathrm{Log}(2z - 3j)$

Solution. Recall that $\mathrm{Log}(z)$ is differentiable everywhere except along the negative real axis. Thus the function $f(z) = \mathrm{Log}(2(z - 3j/2))$ is differentiable everywhere except along the line where $y = 3/2$ and $x \le 0$. (See the diagram below. The red line indicates where the function $f(z)$ is not defined or not differentiable.)

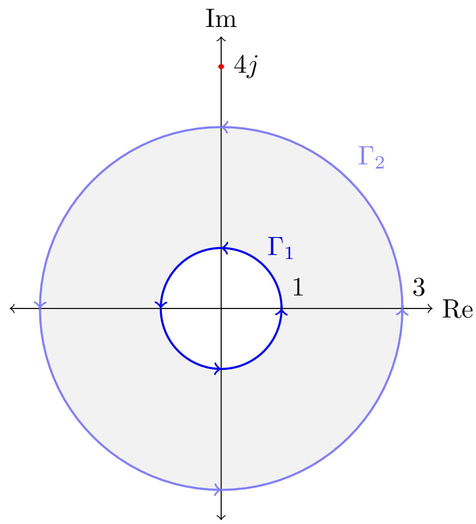


The function is differentiable everywhere on the contour Γ and the region inside of it, so we may use the Cauchy-Goursat theorem to conclude that $\oint_\Gamma f(z)\,dz = 0$.

5. Let Γ_1 be the circle $|z| = 1$ and Γ_2 be the circle $|z| = 3$, each oriented counter clockwise. For which of the following functions does the equality $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ hold?

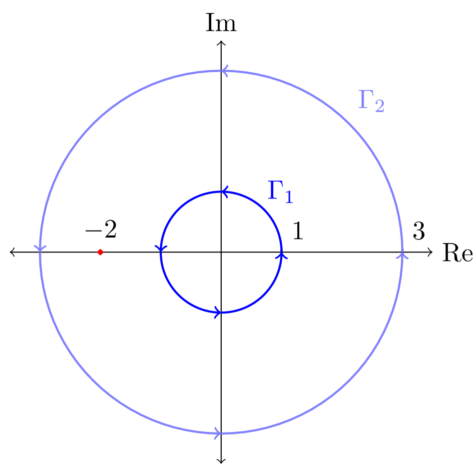
(a) $f(z) = \dfrac{1}{z - 4j}$

Solution. $f(z)$ is not differentiable at $z = 4j$, which is outside of the larger circle Γ_2. This means f is differentiable on Γ_1, Γ_2 and the region in between them, and so the integrals are equal and both have the value zero by the Cauchy-Goursat theorem.



(b) $f(z) = \dfrac{z}{z+2}$

Solution. Note that f is differentiable everywhere except at $z = -2$. In this case, we have that $\displaystyle\int_{\Gamma_1} f(z)\,dz = 0$ since f is differentiable everywhere inside Γ_1 by the Cauchy-Goursat theorem.

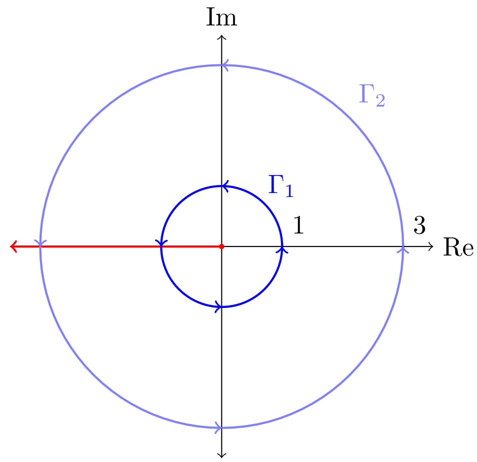


To compute the integral over Γ_2, we can use Cauchy's integral formula by setting $g(z) = z$ to find that

$$\oint_{\Gamma_2} \frac{z}{z+2}\,dz = \oint_{\Gamma_2} \frac{g(z)}{z+2}\,dz = 2\pi j\, g(-2) = -4\pi j.$$

(c) $f(z) = \operatorname{Log} z$

Solution. For both integrals, we have the problem that $\operatorname{Log}(z)$ is not differentiable on the negative real axis.



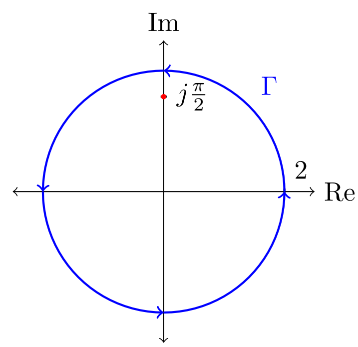
As in problem 5(a), we see that $\int_{\Gamma_1} \operatorname{Log}(z)\, dz = -2\pi j$. Analogously, we can find that $\int_{\Gamma_2} \operatorname{Log}(z)\, dz = -6\pi j$. And thus the integrals are not equal.

6. Use the Cauchy Integral Formula to evaluate the following integrals, where Γ is the circle $|z| = 2$.

(a) $\displaystyle\int_\Gamma \frac{e^z}{z - j\frac{\pi}{2}}\, dz$

Solution. The integrand is differentiable everywhere except at $z_0 = j\frac{\pi}{2}$, which lies inside Γ. We can identify $f(z) = e^z$, and write

$$\int_\Gamma \frac{e^z}{z - j\frac{\pi}{2}}\, dz = 2\pi j\, f\left(j\frac{\pi}{2}\right) = 2\pi j e^{j\frac{\pi}{2}} = 2\pi j(j) = -2\pi.$$

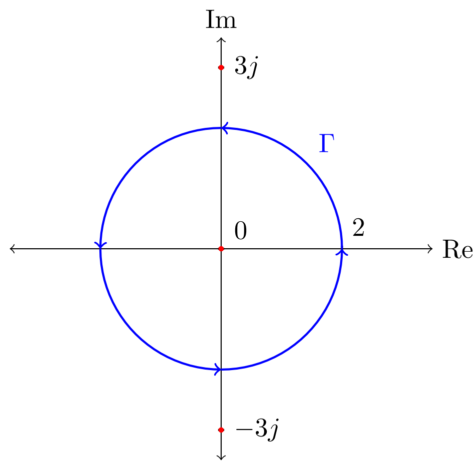


(b) $\displaystyle\int_\Gamma \frac{z^2+1}{z(z^2+9)}\, dz$

Solution. The integrand is differentiable everywhere except at the points $z = 0$ and $z = \pm 3j$. However, the singular points $z = \pm 3j$ lie outside of Γ, thus we can take $f(z) = \frac{z^2+1}{z^2+9}$

as our differentiable function in Γ. Hence, the Cauchy integral formula gives

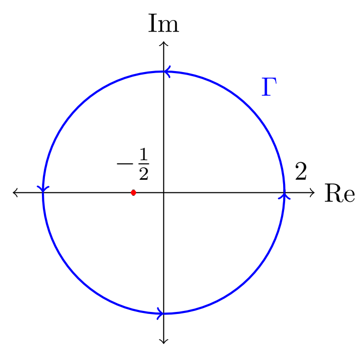
$$\int_\Gamma \frac{z^2+1}{z(z^2+9)}\,dz = 2\pi j f(0) = \frac{2}{9}\pi j.$$



(c) $\displaystyle\oint_\Gamma \frac{z}{2z+1}\,dz$

Solution. The integrand is differentiable everywhere except at $z=-\frac{1}{2}$ which is inside Γ. We need to slightly re-write the integrand to use the Cauchy integral formula: $\frac{z}{2z+1} = \frac{z/2}{z+\frac{1}{2}}$. Thus, our differentiable function in Γ is $f(z)=\frac{z}{2}$. The formula gives

$$\oint_\Gamma \frac{z}{2z+1}\,dz = 2\pi j\, f\left(-\frac{1}{2}\right) = -\frac{\pi}{2}j.$$

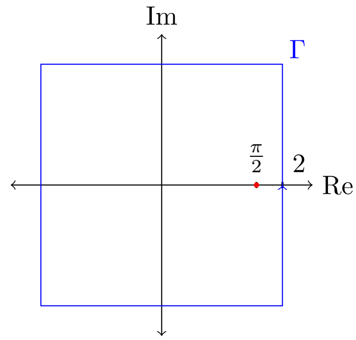


7. Use the generalized Cauchy Integral Formula to evaluate the following, where Γ is the square with edges on $x=\pm 2$ and $y=\pm 2$.

(a) $\displaystyle\oint_\Gamma \frac{\tan(\frac{z}{2})}{(z-\frac{\pi}{2})^2}\,dz$

Solution. $z = \frac{\pi}{2}$ is the singular point, which lies inside of C. With $f(z) = \tan(\frac{z}{2})$, we have $f'(z) = \frac{1}{2}\sec^2(\frac{z}{2})$, and the integral is given by

$$\int_\Gamma \frac{\tan(\frac{z}{2})}{(z-\frac{\pi}{2})^2}\,dz = 2\pi j f'(\pi/2) = \pi j \sec^2(\pi/4) = 2\pi j.$$



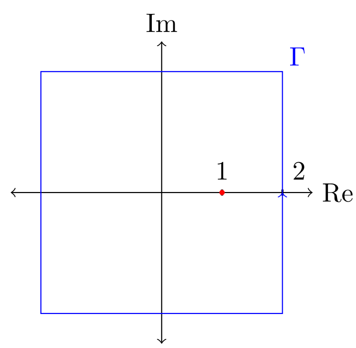
(b) $\displaystyle\int_\Gamma \frac{ze^z}{(z-1)^4}\,dz$

Solution. Note that $z = 1$ is the singular point of the integrand. Setting $f(z) = ze^z$, we need to find $f^{(3)}(z)$. Using the product rule

$$f'(z) = (z+1)e^z, \quad f''(z) = (z+2)e^z, \quad f^{(3)}(z) = (z+3)e^z.$$

Our integral is therefore

$$\int_\Gamma \frac{ze^z}{(z-1)^4}\,dz = \frac{2\pi j}{3!} f^{(3)}(1) = \frac{4e\pi}{3} j.$$

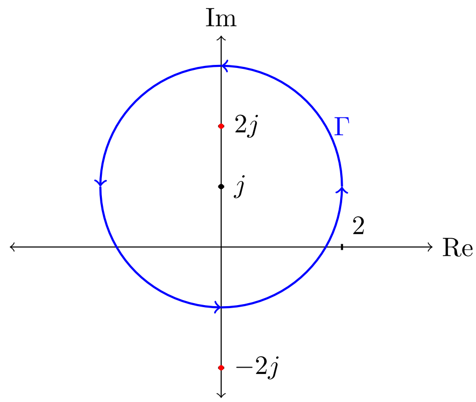


8. Let Γ be the circle $|z-j| = 2$. Evaluate

(a) $\displaystyle\int_\Gamma \frac{1}{z^2+4}\,dz$

Solution. There are two singular points which lie at $z = \pm 2j$. Notice that $z = 2j$ lies within C, but $z = -2j$ does not. We can therefore express the integrand as $\frac{1}{z^2+4} = \frac{1}{(z+2j)(z-2j)}$, and make use of the differentiable function $f(z) = \frac{1}{z+2j}$. By the Cauchy integral formula, we have

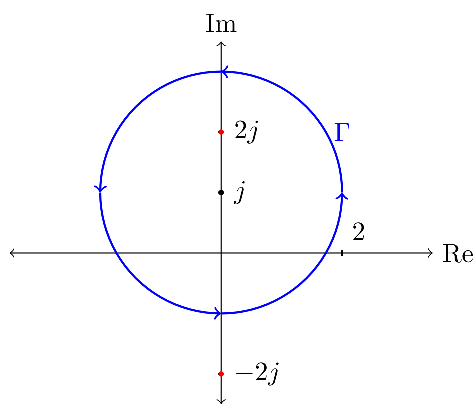
$$\int_\Gamma \frac{1}{z^2+4}\,dz = \int_\Gamma \frac{1}{(z+2j)(z-2j)}\,dz = 2\pi j f(2j) = 2\pi j \cdot \frac{1}{4j} = \frac{\pi}{2}.$$



(b) $\displaystyle\int_\Gamma \frac{1}{(z^2+4)^2}\,dz$

Solution. Similar to the previous part, we write $\frac{1}{(z^2+4)^2} = \frac{1}{(z+2j)^2(z-2j)^2}$. We therefore define the function $f(z) = \frac{1}{(z+2j)^2}$, which is differentiable everywhere inside Γ. Note also that $f'(z) = -\frac{2}{(z+2j)^3}$. We can evaluate the integral as

$$\int_\Gamma \frac{1}{(z^2+4)^2}\,dz = \int_\Gamma \frac{1}{(z+2j)^2(z-2j)^2}\,dz = 2\pi j f'(2j) = 2\pi j \cdot -\frac{2}{(4j)^3} = \frac{\pi}{16}.$$



9. Let Γ be the unit circle parameterized by $z = e^{j\theta}$ for $-\pi \le \theta \le \pi$. Show that for any real constant a,

$$\int_\Gamma \frac{e^{az}}{z}\,dz = 2\pi j$$

Solution. The integrand is differentiable everywhere except $z = 0$, which is the only singular point. This point is inside Γ, so we apply the Cauchy integral formula with $f(z) = e^{az}$ to find that

$$\oint_\Gamma \frac{e^{az}}{z}\, dz = 2\pi j f(0) = 2\pi j.$$

Note. We can use the above integral to derive the following formula:

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta) = \pi.$$

Indeed, using the parameterization $\gamma(\theta) = e^{j\theta}$, we have $\gamma'(\theta) = je^{j\theta}$. Substituting into the integral, we find

$$\begin{aligned}\oint_\Gamma \frac{e^{az}}{z}\, dz = \int_{-\pi}^{\pi} \frac{e^{ae^{j\theta}}}{e^{j\theta}} je^{j\theta}\, d\theta = j\int_{-\pi}^{\pi} e^{a\cos\theta + ja\sin\theta}\, d\theta \\ = j\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ja\sin\theta}\, d\theta \\ = j\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + j\sin(a\sin\theta))\, d\theta\end{aligned}$$

We can split this into two integrals, one real and one imaginary, as follows:

$$\int_\Gamma \frac{e^{az}}{z}\, dz = -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta))\, d\theta + j\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta))\, d\theta$$

And since we know this is $2\pi j$, we look at the imaginary parts to find that

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta))\, d\theta = 2\pi.$$

Finally, noticing the integral is an even function, we obtain the desired result

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\, d\theta = \pi.$$

It is worth mentioning that this is a *real* integral. We have used methods of contour integration of complex functions to compute a real integral. **Interesting!**

10. Let Γ be the circle $|z| = 2$. Evaluate $\displaystyle\int_\Gamma \frac{\sin z}{z^2+1}\, dz$.

Solution. The integrand has singular points at $z = \pm j$, which both lie inside Γ. One option is to split this up into two separate terms to make use of Cauchy's integral formula. Leaving the $\sin z$ aside, we can use partial fractions to re-write

$$\frac{1}{z^2+1} = \frac{A}{z-j} + \frac{B}{z+j} = \frac{A(z+j) + B(z-j)}{z^2+1} \qquad \longrightarrow \qquad 1 = A(z+j) + B(z-j).$$

Putting in $z = j$, we find that $1 = 2Aj$ or $A = \frac{1}{2j} = -\frac{j}{2}$. With $z = -j$, we see that $1 = -2Bj$ and thus $B = -\frac{1}{2j} = \frac{j}{2}$. We can therefore express $\frac{1}{z^2+1} = \frac{j}{2}\left(-\frac{1}{z-j} + \frac{1}{z+j}\right)$.

The integral becomes

$$\int_\Gamma \frac{\sin z}{z^2+1}\,dz = \frac{j}{2}\int_\Gamma \left(\frac{-\sin z}{z-j} + \frac{\sin z}{z+j}\right)\,dz = \frac{j}{2}\left[\int_\Gamma \frac{-\sin z}{z-j}\,dz + \int_\Gamma \frac{\sin z}{z+j}\,dz\right] \qquad (*)$$

Now the Cauchy integral formula applies to each. In the first, we have $f(z) = -\sin z$, so

$$\int_\Gamma \frac{-\sin z}{z-j}\,dz = 2\pi j f(j) = -2\pi j \sin j.$$

Meanwhile, in the second integral, we use the function $f(z) = \sin z$ so that

$$\int_\Gamma \frac{\sin z}{z+j}\,dz = 2\pi j f(-j) = 2\pi j \sin(-j) = -2\pi j \sin j.$$

Thus, $(*)$ simplifies to

$$\frac{j}{2}(-2\pi j \sin j - 2\pi j \sin j) = 2\pi \sin j = 2\pi j \sinh 1 = j\pi\left(e - \frac{1}{e}\right),$$

since $\sin j = j \sinh 1$.