ECE 206 Fall 2019
Practice Problems Weeks 12 & 13

1. (a) Find the Taylor series expansions for $\sinh z$ and $\cosh z$ about $z_0 = 0$ by starting with Taylor series for $\sin z$ and $\cos z$,

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

and using the identities $\sin(jz) = j \sinh z$ and $\cos(jz) = \cosh z$.

   (b) Find the Taylor series expansion of $f(z) = \frac{z}{z^4+9}$ about $z_0 = 0$. Give the region of validity.

2. Find all possible Laurent series expansions for $f(z) = \frac{1}{z+j}$ about $z_0 = 0$ and give the region of validity for each.

3. Find the first three nonzero terms of the Laurent series for each of the following mappings that is valid in the given regions.

   (a) $\dfrac{1}{\sin z}$ in the region where $0 < |z| < \pi$.

   (b) $\dfrac{1}{\cos z}$ in the region where $0 < \left|z - \frac{\pi}{2}\right| < \pi$.

   (c) e^{-1/z^3} in the region where $0 < |z| < \infty$.

4. Find the Laurent series for $f(z) = \dfrac{1}{z(z-1)^2}$ about

   (a) $z = 0$ that is valid for $0 < |z| < 1$

   (b) $z = 0$ that is valid for $|z| > 1$

   (c) $z = 1$ that is valid for $0 < |z - 1| < 1$

   (d) $z = 1$ that is valid for $|z - 1| > 1$

5. For each function, find and identify the type of each singularity (i.e., removable, pole of order m, or isolated singularity) and find the residue there.

   (a) $f(z) = \dfrac{z^2+2}{z-1}$

   (b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

   (c) $f(z) = \dfrac{1}{z^3(z-1)^2(z-2)}$

6. For each of the following, show that the singular point is a pole. Determine the order of the pole as well as the residue.

   (a) $f(z) = \dfrac{1 - \cosh z}{z^3}$

   (b) $f(z) = \dfrac{1 - e^{2z}}{z^4}$

(c) $f(z) = \dfrac{e^{2z}}{(z-1)^2}$

7. Find the residue at $z = 0$ of the following functions by writing out the Laurent series.

   (a) $f(z) = \dfrac{1}{z + z^2}$

   (b) $f(z) = z \cos\left(\dfrac{1}{z}\right)$

   (c) $f(z) = \dfrac{z - \sin z}{z^3}$

   (d) $f(z) = \dfrac{\sinh z}{z^4(1 - z^2)}$

8. Consider the function $f(z) = \frac{\sin z}{(1-\cos z)^2}$.

   (a) Show that the point $z = 0$ is a pole of the function $f(z)$ and find the order of the pole.

   (b) If m is the order of the pole, find the coefficient c_{-m} of $\frac{1}{z^m}$ term in the Laurent series expansion of f at $z = 0$.

   (c) What is the residue of $f(z)$ at $z = 0$?

9. Use the method of ML-estimation to evaluate the following improper integrals by integrating around a semicircular contour in the upper half-plane.

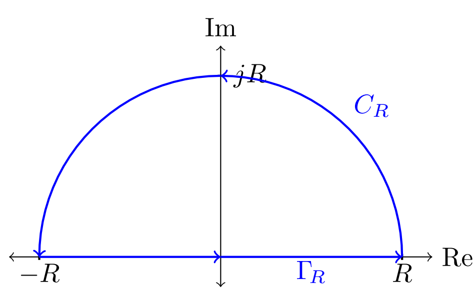
   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4x + 5}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2)}\, dx$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4 + a^4}\, dx$, where $a > 0$ is a constant.

10. Let a be a positive real number ($a > 0$). For each $R > 1$, let Γ_R be the part of the real axis from $-R$ to R and C_R be the semicircular contour of radius R in the upper half plane going counterclockwise (see the figure below) such that $\Gamma_R \cup C_R$ is a closed contour.



   (a) Evaluate the integral $\displaystyle\oint_{\Gamma_R \cup C_R} \frac{e^{jaz}}{z^2+1}\, dz$.

   (b) Use the ML-estimation technique and your answer from part (a) to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{jax}}{x^2+1}\, dx$.

(c) From the fact that $\operatorname{Re}\left(e^{jax}\right) = \cos ax$, use your answer from part (b) to conclude that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \pi e^{-a}$$

11. Use contour integration to evaluate $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos\theta}\, d\theta$.

    (*We didn't cover this in lecture, so you may ignore it.*)

12. Show using contour integration that $\displaystyle\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}\, d\theta = \frac{2\pi}{1-a^2}$, where $|a| < 1$ is a constant.

    (*We didn't cover this in lecture, so you may ignore it.*)

13. Use contour integrals to compute the inverse Fourier transforms of the following.

    (a) $F(\omega) = \dfrac{1}{\omega^2 - 3j\omega - 2}$

    (b) $F(\omega) = \dfrac{1}{(2-j\omega)^2}$