# ECE 206 Fall 2019
# Practice Problems Weeks 12 & 13
# **Solutions**

1. (a) Find the Taylor series expansions for $\sinh z$ and $\cosh z$ about $z_0 = 0$ by starting with Taylor series for $\sin z$ and $\cos z$,

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

and using the identities $\sin(jz) = j \sinh z$ and $\cos(jz) = \cosh z$.

*Solution.* Start from the Taylor series for $\sin z$ above and use the identity $\sin(jz) = j\sinh z$ to obtain

$$\sinh z = -j\sin(jz) = -j\sum_{n=0}^{\infty}(-1)^n \frac{(jz)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} -j^2 \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

where we use the facts that $j^{2n+1} = j^{2n} j = (j^2)^n j = (-1)^n j$ and $(-1)^n(-1)^n = (-1)^{2n} = 1$ for any integer $n$. Similarly, we have

$$\cosh z = \cos(jz) = \sum_{n=0}^{\infty} (-1)^n \frac{(jz)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

(b) Find the Taylor series expansion of $f(z) = \frac{z}{z^4+9}$ about $z_0 = 0$. Give the region of validity.

*Solution.* We first rewrite $f(z)$ in terms of functions with known Taylor series as

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\frac{1}{1+\frac{z^4}{9}}$$

and use the known Taylor series for functions of the form $1/(1-w)$ where we set $w = -z^4/9$ to find that

$$\frac{1}{1+\frac{z^4}{9}} = \sum_{n=0}^{\infty}\left(\frac{-z^4}{9}\right)^n, \qquad \text{which is valid in the region where } \left|\frac{z^4}{9}\right| < 1.$$

Multiplying through, we obtain

$$f(z) = \frac{z}{9}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} z^{4n+1}$$

which is valid in the region where $|z| < \sqrt{3}$.

2. Find all possible Laurent series expansions for $f(z) = \frac{1}{z+j}$ about $z_0 = 0$ and give the region of validity for each.

3. Find the first three nonzero terms of the Laurent series for each of the following mappings that is valid in the given regions.

(a) $\dfrac{1}{\sin z}$ in the region where $0 < |z| < \pi$.

*Solution.* Using the known Taylor series expansion for $\sin z$, we find

$$\begin{aligned}
\frac{1}{\sin z} &= \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots} \\
&= \frac{1}{z}\frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right)} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right)^n \\
&= \frac{1}{z}\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right)^2 + \cdots\right) \\
&= \frac{1}{z}\left(1 + \frac{z^2}{3!} + \left(\frac{1}{3!^2} - \frac{1}{5!}\right)z^4 + \cdots\right) \\
&= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots,
\end{aligned}$$

where we make use of the Taylor series expansion for $1/(1-w)$ and multiply out.

(b) $\dfrac{1}{\cos z}$ in the region where $0 < \left|z - \frac{\pi}{2}\right| < \pi$.

*Solution.* We first rewrite $f(z)$ as a funciton of $z - \pi/2$ by noting that

$$\cos z = \cos\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right) = -\sin\left(z - \frac{\pi}{2}\right).$$

Now, making use of the Laurent series for $1/\sin w$ that we found in part (a), we have

$$\begin{aligned}
\frac{1}{\cos z} &= -\frac{1}{\sin\left(z - \frac{\pi}{2}\right)} \\
&= -\frac{1}{z - \frac{\pi}{2}} - \frac{1}{6}\left(z - \frac{\pi}{2}\right) - \frac{7}{360}\left(z - \frac{\pi}{2}\right)^3 + \cdots.
\end{aligned}$$

(c) $e^{-1/z^3}$ in the region where $0 < |z| < \infty$.

*Solution.* Here we simply set $w = -1/z^3$ and use the known Taylor series for $e^w$ to find that

$$\begin{aligned}
e^{-1/z^3} &= 1 - \frac{1}{z^3} + \frac{1}{2!}\left(-\frac{1}{z^3}\right)^2 + \cdots \\
&= 1 - \frac{1}{z^3} + \frac{1}{2}\frac{1}{z^6} - \cdots
\end{aligned}$$

4. Find the Laurent series for $f(z) = \dfrac{1}{z(z-1)^2}$ about

(a) $z = 0$ that is valid for $0 < |z| < 1$

*Solution.* We first start with the fact that

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}.$$

Since we know the Taylor series for $1/(1-z)$, this means we can find the Taylor series of $1/(1-z)^2$ by

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} nz^{n-1}, \tag{$*$}$$

which is valid only for $|z| < 1$. Thus the Laurent series for $f$ is given by

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} nz^{n-1} = \frac{1}{z} + 2 + 3z + 4z^2 + \cdots = \frac{1}{z} + \sum_{n=0}^{\infty}(n+2)z^n,$$

which is valid for $0 < |z| < 1$.

(b) $z = 0$ that is valid for $|z| > 1$

*Solution.* We first rearrange the terms to write $f(z)$ as a function of $1/z$ and find that

$$\frac{1}{(z-1)^2} = \frac{1}{z^2} \cdot \frac{1}{\left(1 - \frac{1}{z}\right)^2}.$$

We now make use of the Laurent series from ($*$) in part (a) above to find that

$$\frac{1}{\left(1-\frac{1}{z}\right)^2} = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^{n-1}$$

which is valid for $0 < |1/z| < 1$, or equivalently $1 < |z| < \infty$. The desired Laurent series is therefore

$$f(z) = \frac{1}{z^3}\sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^{n-1} = \sum_{n=0}^{\infty}\frac{n}{z^{n+2}} = \sum_{n=0}^{\infty}\frac{n}{z^{n+2}}.$$

(c) $z = 1$ that is valid for $0 < |z-1| < 1$

*Solution.* For this and the next part, we must express $1/z$ in terms of powers of $z - 1$.

We can do so as follows: $\dfrac{1}{z} = \dfrac{1}{1-(1-z)}$. Then

$$\begin{aligned} f(z) = \frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2}\frac{1}{1-(1-z)} &= \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(1-z)^n \\ &= \frac{1}{(z-1)^2}\left(1+(1-z)+(1-z)^2+\dots\right) \\ &= \frac{1}{(z-1)^2}\left(1-(z-1)+(z-1)^2-\dots\right) \\ &= \frac{1}{(z-1)^2} - \frac{1}{z-1} + \sum_{n=0}^{\infty}(-1)^n(z-1)^n. \end{aligned}$$

(d) $z = 1$ that is valid for $|z-1| > 1$

*Solution.*

$$f(z) = \frac{1}{(z-1)^2}\cdot\frac{1}{1-z}\cdot\frac{1}{\frac{1}{1-z}-1} = \frac{1}{(z-1)^3}\cdot\frac{1}{1-\frac{1}{1-z}} = \frac{1}{(z-1)^3}\sum_{n=0}^{\infty}\left(\frac{1}{1-z}\right)^n$$

We can write this as

$$f(z) = \frac{1}{(z-1)^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+3}}$$

5. For each function, find and identify the type of each singularity (i.e., removable, pole of order $m$, or isolated singularity) and find the residue there.

(a) $f(z) = \dfrac{z^2+2}{z-1}$

*Solution.* The pole of $f$ is a pole of order 1 at $z = 1$. The residue here is

$$\mathrm{Res}\,(f,1) = \lim_{z\to1}[(z-1)f(z)] = \lim_{z\to1} z^2+2 = 3$$

(b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

*Solution.* There is a pole of order 3 at $z = -1/2$. We may re-write $f(z)$ as a function of $z + 1/2$ as

$$f(z) = \frac{z^3/8}{(z+\frac{1}{2})^3}$$

The residue is given by

$$\mathrm{Res}\left(f, -\frac{1}{2}\right) = \frac{1}{2!}\lim_{z\to-\frac{1}{2}}\left[\frac{d^2}{dz^2}\left(\frac{z^3}{8}\right)\right] = \frac{1}{2}\lim_{z\to-\frac{1}{2}}\frac{3z}{4} = -\frac{3}{16}.$$

(c) $f(z) = \dfrac{1}{z^3(z-1)^2(z-2)}$

*Solution.* Here we have a pole of order 3 at $z = 0$, a pole of order 2 at $z = 1$, and a simple pole at $z = 2$. The residue at $z = 2$ is

$$\text{Res}(f, 2) = \lim_{z \to 2} \frac{1}{z^3(z-1)^2} = \frac{1}{8}.$$

At $z = 1$, the residue is

$$\text{Res}(f, 1) = \lim_{z \to 1} \left[ \frac{d}{dz} \left( \frac{1}{z^3(z-2)} \right) \right] = \lim_{z \to 1} \frac{d}{dz} (z^{-3}(z-2)^{-1}) = \cdots = 2.$$

At $z = 0$, the residue is

$$\text{Res}(f, 0) = \frac{1}{2!} \lim_{z \to 0} \left[ \frac{d^2}{dz^2} \left( \frac{1}{(z-1)^2(z-2)} \right) \right] = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} ((z-1)^{-2}(z-2)^{-1}) = \cdots = -\frac{17}{8}.$$

6. For each of the following, show that the singular point is a pole. Determine the order of the pole as well as the residue.

(a) $f(z) = \dfrac{1 - \cosh z}{z^3}$

*Solution.* Writing out the Laurent series, we have

$$f(z) = \frac{1}{z^3} \left( 1 - \left( 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \right) \right) = -\frac{1}{2z} - \frac{z}{4!} - \dots$$

and this expansion is valid for all $z \neq 0$. Thus, $f$ has a pole of order 1 at $z = 0$ and the residue at this point is is $-\frac{1}{2}$.

(b) $f(z) = \dfrac{1 - e^{2z}}{z^4}$

*Solution.* The Laurent expansion is

$$f(z) = \frac{1}{z^4} \left( 1 - \left( 1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \dots \right) \right) = -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \dots$$

which is valid for all $z \neq 0$. Thus, $f$ has a pole of order 3 at $z = 0$, with residue $-\frac{4}{3}$.

(c) $f(z) = \dfrac{e^{2z}}{(z-1)^2}$

*Solution.* First, re-write $f$ with a shift, then use the Taylor series for the exponential to expand $f$ as

$$f(z) = \frac{e^{2(z-1+1)}}{(z-1)^2} = \frac{e^2}{(z-1)^2} \left( 1 + 2(z-1) + 4\frac{(z-1)^2}{2!} + \dots \right) = \frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1} + 2e^2 + \dots$$

which is valid for all $z \neq 1$. Thus $f$ has a pole of order 2 at $z = 1$, with residue $2e^2$.

7. Find the residue at $z = 0$ of the following functions by writing out the Laurent series.

(a) $f(z) = \dfrac{1}{z + z^2}$

*Solution.* We can expand this as

$$f(z) = \frac{1}{z}\frac{1}{1-(-z)} = \frac{1}{z}\left(1 + (-1)z + (-1)^2 z^2 + (-1)^3 z^3 + \cdots\right)$$
$$= \frac{1}{z} - 1 + z - z^2 + \cdots = \frac{1}{z} - \sum_{n=0}^{\infty} (-1)^n z^n.$$

This expansion is valid for all $0 < |z| < 1$. Hence $z = 0$ is a pole of order 1 of $f$ and the desired residue is $\mathrm{Res}(f, 0) = 1$.

(b) $f(z) = z \cos\left(\dfrac{1}{z}\right)$

*Solution.* Using the known Taylor series for cos, we can expand this as

$$f(z) = z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \dots\right) = z - \frac{1}{2!z} + \dots$$

This expansion is valid for all $z \neq 0$. Hence $z = 0$ is an essential singularity of $f$ and the residue of $f$ at $z = 0$ is $-\frac{1}{2}$.

(c) $f(z) = \dfrac{z - \sin z}{z^3}$

*Solution.* Using the known Taylor series for sin, we can expand this as

$$f(z) = \frac{1}{z^3}(z - \sin z) = \frac{1}{z^3}\left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right)$$
$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)!} z^{2n},$$

which has no principal part and is valid for all $z \in \mathbb{C}$. Hence the point $z = 0$ is a removeable singularity of $f$ and $\mathrm{Res}(f, 0) = 0$. In fact, with repeated application of L'hopital's rule, we have

$$\lim_{z\to 0} \frac{z - \sin z}{z^3} = \lim_{z\to 0} \frac{\cos z}{3z^2} = \lim_{z\to 0} \frac{-\sin z}{6z} \lim_{z\to 0} \frac{-\cos z}{6} = -\frac{1}{6}.$$

(d) $f(z) = \dfrac{\sinh z}{z^4(1 - z^2)}$

*Solution.* Using the known expansion for sinh, we have

$$f(z) = \frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4} \cdot \sinh z \cdot \frac{1}{1-z^2} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)(1 + z^2 + z^4 + \dots)$$
$$= \frac{1}{z^4}\left(z + \left(\frac{1}{3!} + 1\right) z^3 + \dots\right)$$
$$= \frac{1}{z^3} + \frac{7}{6z} + \dots$$

and this expansion is valid for all $0 < |z| < 1$. Hence the function $f$ has a pole of order 3 at the point $z = 0$ and the the residue at $z = 0$ is $\frac{7}{6}$.

8. Consider the function $f(z) = \frac{\sin z}{(1-\cos z)^2}$.

   (a) Show that the point $z = 0$ is a pole of the function $f(z)$ and find the order of the pole.

*Solution.* Here we will use Theorem 24.4.1 from the textbook, which we will restate here.

**Theorem.** *(Nth-order pole)*
If $p(z)$ and $q(z)$ have zeros of order $P$ and $Q$, respectively, at $z = z_0$, then $f(z) = \frac{p(z)}{q(z)}$ has a pole or order $N = Q - P$ there if $Q > P$, and is analytic there if $Q \leq P$.

Recall that a function $g(z)$ has a *zero of order* $n$ at $z = z_0$ if

$$g(z_0) = g'(z_0) = g''(x_0) \cdots = g^{(n-1)}(z_0) = 0 \qquad \text{but} \qquad g^{(n)}(z_0) \neq 0.$$

We can use the Theorem by taking $p(z) = \sin z$ and $q(z) = (1 - \cos z)^2$. Note that $p(z) = \sin z$ has a first order zero at $z = 0$ since $\sin(0) = 0$ but $\cos(0) = 1 \neq 0$. However, $q(z) = (1-\cos z)^2$ has a fourth order zero at $z = 0$ since $g(0) = g'(0) = g''(0) = g'''(0) = 0$ but $g^{(4)} = 6 \neq 1$. Hence $z = 0$ is a pole of order 3.

   (b) If $m$ is the order of the pole, find the coefficient $c_{-m}$ of $\frac{1}{z^m}$ term in the Laurent series expansion of $f$ at $z = 0$.

*Solution.* Note that 0 is a third order pole, so we may compute $c_{-3}$ using successive applications of L'hopital's rule as

$$\lim_{z\to 0}\left[z^3 f(z)\right] = \lim_{z\to 0} \frac{z^3 \sin z}{(1-\cos z)^2}$$
$$= \lim_{z\to 0} \frac{3z^2 \sin z + z^3 \cos z}{2(1-\cos z)\sin z}$$
$$= \lim_{z\to 0} \frac{6z \sin z + 6z^2 \cos z - z^3 \sin z}{2(1-\cos z)\sin z}$$
$$= \cdots$$
$$= 4.$$

   (c) What is the residue of $f(z)$ at $z = 0$?

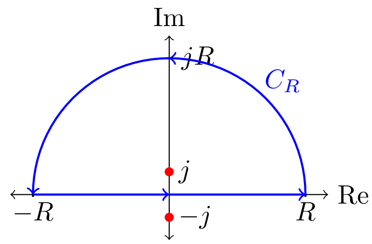
9. Use the method of ML-estimation to evaluate the following improper integrals by integrating around a semicircular contour in the upper half-plane.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

*Solution.* For each $R > 1$, consider the simple closed contour consisting of the semicircle of radius $R$ centered at the origin that lies in the upper half-plane and the part of the real axis from $-R$ to $R$ (pictured below). Consider the function

$$f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z-j)^3}\frac{1}{(z+j)^3},$$

which is analytic everywhere except at the singularities $z = \pm j$.




By choosing the function $g(z) = \frac{1}{(z+j)^3}$ and noting that $g'(z) = \frac{-3}{(z+j)^4}$ and $g''(z) = \frac{12}{(z+j)^5}$, we can use the Cauchy integral formula to see that

$$\oint_{\Gamma_R \cup C_R} \frac{1}{(z^2+1)^3}\,dz = \oint_{\Gamma_R \cup C_R} \frac{g(z)}{(z-j)^3}\,dz = \frac{2\pi j}{2!}g''(j) = \pi j \frac{12}{(j+j)^5} = \frac{3\pi}{8},$$

which holds for any $R > 1$. Note that $\left|z^2+1\right| \geq |z|^2 - 1$ for any $z \in \mathbb{C}$. Thus, for any complex number $z = Re^{j\theta}$ on the semicircular arc of radius $R > 1$, we have $|z| = R$ and

$$\left|\frac{1}{(z^2+1)^3}\right| = \frac{1}{\left|z^2+1\right|^3} \leq \frac{1}{\left(|z|^2-1\right)^3} = \frac{1}{(R^2-1)^3}.$$

Thus, by the $ML$-theorem, it holds that

$$\left|\int_{C_R} \frac{1}{(z^2+1)^3}\,dz\right| \leq \frac{\pi R}{(R^2-1)^3},$$

where $C_R$ is the contour of the semicircular arc from $R$ to $-R$ going counterclockwise in the upper half-plane. Moreover, we have that

$$\lim_{R\to\infty}\left(\int_{C_R} \frac{1}{(z^2+1)^3}\,dz\right) = 0 \qquad \text{since} \qquad \lim_{R\to\infty} \frac{\pi R}{(R^2-1)^3} = 0.$$

Finally, we compute the desired integral as

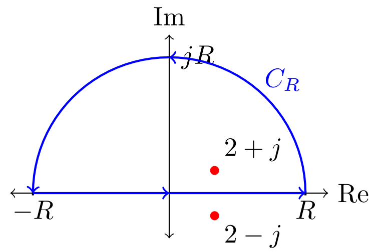
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} &= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+1)^3} \\ &= \lim_{R\to\infty}\left(\oint_{\Gamma_R\cup C_R} \frac{1}{(z^2+1)^3}\,dz - \int_{C_R} \frac{1}{(z^2+1)^3}\,dz\right) \\ &= \frac{3\pi}{8}.\end{aligned}$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4x + 5}$

*Solution.* We use the same contours as in the previous problem. Consider the function

$$f(z) = \frac{1}{z^2 - 4z + 5} = \frac{1}{(z-2-j)(z-2+j)},$$

which is analytic everywhere except at the singularities $z = 2 \pm j$.




By choosing the function $g(z) = \frac{1}{z-2+j}$, we use the Cauchy integral formula to see that

$$\oint_{\Gamma_R \cup C_R} \frac{1}{z^2 - 4z + 5}\, dz = \oint_{\Gamma_R \cup C_R} \frac{g(z)}{z - 2 - j}\, dz = 2\pi j\, g(2+j) = 2\pi j \frac{1}{2j} = \pi,$$

which holds for any $R > \sqrt{5}$. Note that $\left|z^2 - 4z + 5\right| \geq |z|^2 - 4\,|z| - 5$ for any $z \in \mathbb{C}$. Thus, for any complex number $z = Re^{j\theta}$ on a semicircular arc of radius $R > 5$, we have $|z| = R$ and

$$\left|\frac{1}{z^2 - 4z + 5}\right| = \frac{1}{|z^2 - 4z + 5|} \leq \frac{1}{|z|^2 - 4\,|z| - 5} = \frac{1}{R^2 - 4R - 5}.$$

Thus, by the $ML$-theorem, it holds that

$$\left|\int_{C_R} \frac{1}{z^2 - 4z + 5}\, dz\right| \leq \frac{\pi R}{R^2 - 4R - 5},$$

where $C_R$ is the contour of the semicircular arc from $R$ to $-R$ going counterclockwise in the upper half-plane. Moreover, we have that

$$\lim_{R \to 0} \left(\int_{C_R} \frac{1}{z^2 - 4z + 5}\, dz\right) = 0 \qquad \text{since} \qquad \lim_{R \to \infty} \frac{\pi R}{R^2 - 4R - 5} = 0.$$

Finally, we compute the desired integral as

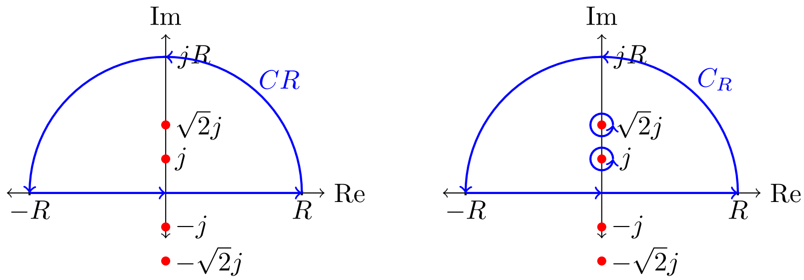
$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4x + 5} &= \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2 - 4x + 5} \\
&= \lim_{R \to \infty} \left(\oint_{\Gamma_R \cup C_R} \frac{1}{z^2 - 4z + 5}\, dz - \int_{C_R} \frac{1}{z^2 - 4z + 5}\, dz\right) \\
&= \pi.
\end{aligned}$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2)}\, dx$

*Solution.* We use the same contours as in the parts (a) and (b). Consider the function

$$f(z) = \frac{z^2}{(z^2+1)(z^2+2)} = \frac{z^2}{(z-j)(z+j)(z-\sqrt{2}j)(z+\sqrt{2}j)},$$

which is analytic everywhere except at the singularities $z = \pm j$ and $z = \pm\sqrt{2}j$.




By the Residue Theorem, for any $R > \sqrt{2}$ we have that

$$\oint_{\Gamma_R \cup C_R} \frac{z^2}{(z^2+1)(z^2+2)}\, dz = 2\pi j \operatorname{Res}(f, j) + 2\pi j \operatorname{Res}(f, \sqrt{2}j)$$

for any $R > \sqrt{2}$. Note that, for any $z = Re^{j\theta}$ with $R > \sqrt{2}$, we have

$$\left| \frac{z^2}{(z^2+1)(z^2+2)} \right| = \frac{|z|^2}{|z^2+1|\,|z^2+2|} \leq \frac{|z|^2}{(|z|^2-1)(|z|^2-2)} = \frac{R^2}{(R^2-1)(R^2-2)}.$$

Thus, by the $ML$-theorem, it holds that

$$\left| \int_{C_R} \frac{z^2}{(z^2+1)(z^2+2)}\, dz \right| \leq \frac{\pi R^3}{(R^2-1)(R^2-2)},$$

where $C_R$ is the contour of the semicircular arc from $R$ to $-R$ going counterclockwise in the upper half-plane. Moreover, we have that

$$\lim_{R\to\infty} \left( \int_{C_R} \frac{z^2}{(z^2+1)(z^2+2)}\, dz \right) = 0 \qquad \text{since} \qquad \lim_{R\to\infty} \frac{\pi R^3}{(R^2-1)(R^2-2)} = 0.$$

The singularities of $f$ at $z = j$ and $z = \sqrt{2}j$ are poles of order 1. The desired residues can therefore be computed as

$$\operatorname{Res}(f, j) = \frac{(j)^2}{(j+j)(j^2+2)} = \frac{-1}{(2j)(-1+2)} = \frac{j}{2}$$

and

$$\operatorname{Res}(f, \sqrt{2}j) = \frac{(\sqrt{2}j)^2}{((\sqrt{2}j)^2+1)\left(\sqrt{2}j+\sqrt{2}j\right)} = \frac{-2}{(-1)(2\sqrt{2}j)} = -\frac{j}{\sqrt{2}}$$

Finally, we compute the desired integral as

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2)}\, dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+2)}\, dx \\
&= \lim_{R\to\infty} \left( \oint_{\Gamma_R \cup C_R} \frac{z^2}{(z^2+1)(z^2+2)}\, dz - \int_{C_R} \frac{z^2}{(z^2+1)(z^2+2)}\, dz \right) \\
&= 2\pi j \left( \operatorname{Res}(f, j) + \operatorname{Res}(f, \sqrt{2}j) \right) \\
&= 2\pi j \left( \frac{j}{2} - \frac{j}{\sqrt{2}} \right) = \pi(\sqrt{2} - 1)
\end{aligned}$$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx$, where $a>0$ is a constant.

*Solution.* We proceed as in the previous parts, noting that the associated $|f(z)|$ behaves as $\frac{1}{R^4}$ for large $R$, and thus the integral along the semicircular contour goes to zero as $R \longrightarrow \infty$. We note that there are 4 roots to the equation $z^4+a^4=0$ (i.e., there are four solutions to$z^4=-a^4$), which are $z=a(-1)^{1/4}$. Writing $-1=e^{j(\pi+2k\pi)}$ for integers $k\in\mathbb{Z}$, we have the roots are located at

$$z_k = ae^{j\left(\frac{\pi}{4}+n\frac{\pi}{2}\right)}$$

for $k=0,1,2,3$, or

$$z_0 = ae^{j\pi/4}, \quad z_1 = ae^{j3\pi/4}, \quad z_2 = ae^{j5\pi/4}, \quad z_3 = ae^{j7\pi/4}.$$

Only the first two of these lie in the upper half plane. Since each singularity is a pole order 1, and the integrand has the form $f(z)=\frac{1}{z^4+a^4}$, the residues are computed most easily as

$$\operatorname{Res}(f,z_k) = \frac{1}{4z_k^3}.$$

Evaluating the residue at $z_0 = ae^{j\frac{\pi}{4}}$, we have

$$\operatorname{Res}(f,z_0) = \frac{1}{4a^3e^{j3\pi/4}} = \frac{e^{-j3\pi/4}}{4a^3} = \frac{1}{4a^3}\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right).$$

At $z_1 = ae^{j3\pi/4}$, we have

$$\operatorname{Res}(f,z_1) = \frac{1}{4a^3e^{j9\pi/4}} = \frac{e^{-j9\pi/4}}{4a^3} = \frac{e^{-j\pi/4}}{4a^3} = \frac{1}{4a^3}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right).$$

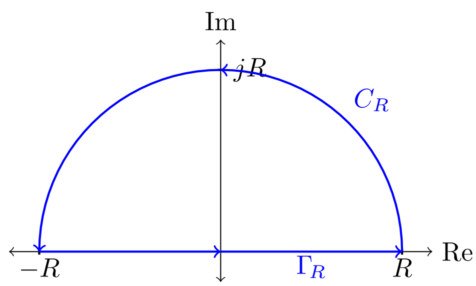
The sum of the residues is thus

$$\frac{1}{4a^3}\left(-j\frac{2}{\sqrt{2}}\right) = -j\frac{1}{2\sqrt{2}a^3}.$$

By the residue theorem, we have

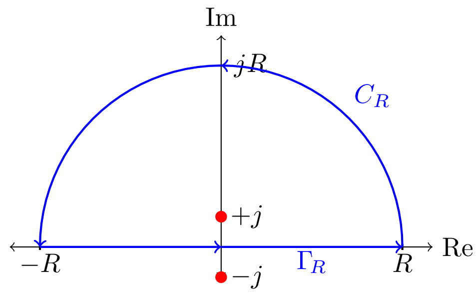
$$\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = 2\pi j\left(-j\frac{1}{2\sqrt{2}}a^3\right) = \frac{\pi}{\sqrt{2}a^3}.$$

10. Let $a$ be a positive real number ($a>0$). For each $R>1$, let $\Gamma_R$ be the part of the real axis from $-R$ to $R$ and $C_R$ be the semicircular contour of radius $R$ in the upper half plane going counterclockwise (see the figure below) such that $\Gamma_R \cup C_R$ is a closed contour.



(a) Evaluate the integral $\oint_{\Gamma_R \cup C_R} \frac{e^{jaz}}{z^2+1}\, dz$.

*Solution.* The integrand $\frac{e^{jaz}}{z^2+1} = \frac{e^{jaz}}{(z-j)(z+j)}$ has poles of order 1 at $z = \pm j$, only one of which lies inside the simple closed contour $\Gamma_R \cup C_R$.




The residue of $f(z) = \frac{e^{jaz}}{z^2+1}$ at $z = j$ is

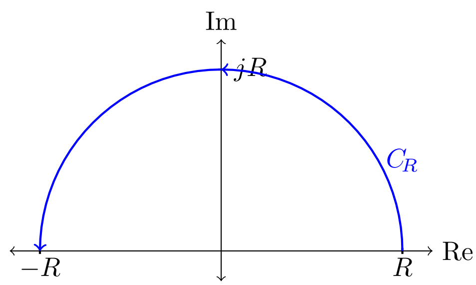
$$\text{Res}(f, j) = \frac{e^{jaj}}{j+j} = -j\frac{e^{-a}}{2}.$$

Using residue theory, we can compute the desired integral as

$$\oint_{\Gamma_R \cup C_R} \frac{e^{jaz}}{z^2+1}\, dz = 2\pi j\, \text{Res}(f, j) = 2\pi j \left(-j\frac{e^{-a}}{2}\right) = \pi e^{-a}.$$

(b) Use the ML-estimation technique and your answer from part (a) to evaluate $\int_{-\infty}^{\infty} \frac{e^{jax}}{x^2+1}\, dx$.

*Solution.* Let $C_R$ denote the semicircular contour of radius $R$ centered at the origin in the upper half plane going counterclockwise from $R$ to $-R$.




For points $z = Re^{j\theta}$ with $\theta \in [0, \pi]$ on this contour, we have

$$\left|e^{jaz}\right| = \left|e^{jaR(\cos\theta + j\sin\theta)}\right| = e^{-aR\sin\theta} \underbrace{\left|e^{jaR\cos\theta}\right|}_{=1} = e^{-aR\sin\theta} \leq 1$$

since $\sin\theta \geq 0$ in the range $\theta \in [0, \pi]$ and thus $-aR\sin\theta \leq 0$ such that $e^{-aR\sin\theta} \leq 1$. Moreover, we have that

$$\left|\frac{e^{jaz}}{z^2+1}\right| = \frac{\left|e^{jaz}\right|}{|z^2+1|} \leq \frac{1}{R^2-1}$$

for all points $z = Re^{j\theta}$ with $\theta \in [0, \pi]$ on this contour. By the $ML$-theorem, we therefore have that

$$\left| \int_{C_R} \frac{e^{jaz}}{z^2+1}\, dz \right| \le \frac{\pi R}{R^2 - 1} \longrightarrow 0$$

in the limit as $R \to \infty$. That follows that

$$\lim_{R\to 0} \int_{C_R} \frac{e^{jaz}}{z^2+1}\, dz = 0$$

ans thus the desired integral can be evaluated as

$$\int_{-\infty}^{\infty} \frac{e^{jax}}{x^2+1}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{jax}}{x^2+1}\, dx = \pi e^{-a} - \left( \lim_{R\to 0} \int_{C_R} \frac{e^{jaz}}{z^2+1}\, dz \right) = \pi e^{-a}.$$

(c) From the fact that $\operatorname{Re}\left(e^{jax}\right) = \cos ax$, use your answer from part (b) to conclude that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \pi e^{-a}$$

*Solution.*

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \int_{-\infty}^{\infty} \frac{\operatorname{Re}(e^{jax})}{x^2+1}\, dx = \operatorname{Re}\left( \int_{-\infty}^{\infty} \frac{e^{jax}}{x^2+1}\, dx \right) = \operatorname{Re}\left(\pi e^{-a}\right) = \pi e^{-a}.$$

11. Use contour integration to evaluate $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5 - 4\cos\theta}\, d\theta.$

(*We didn't cover this in lecture, so you may ignore it.*)

12. Show using contour integration that $\displaystyle\int_0^{2\pi} \frac{1}{1 - 2a\cos\theta + a^2}\, d\theta = \frac{2\pi}{1-a^2}$, where $|a| < 1$ is a constant.

(*We didn't cover this in lecture, so you may ignore it.*)

13. Use contour integrals to compute the inverse Fourier transforms of the following.

(a) $F(\omega) = \dfrac{1}{\omega^2 - 3j\omega - 2}$

*Solution.* We may rewrite $F(\omega)$ as

$$F(\omega) = \frac{1}{(\omega - j)(\omega - 2j)}$$

and observe that this has two simple poles at $z = j$ and $z = 2j$, both of which are located in the upper half plane. Now we integrate

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{j\omega x}}{(\omega - j)(\omega - 2j)}\, d\omega = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{+R} \frac{e^{j\omega x}}{(\omega - j)(\omega - 2j)}\, d\omega.$$

There are two cases to consider: $x > 0$ and $x < 0$.

(i) When $x > 0$, we integrate around the semicircular contour on radus $R$ in the upper half-plane. To find the value of the integral, we only need to compute the residues at the singularities in the upper half-plane. This is

$$f(x) = j\left(\lim_{\omega\to j}\frac{e^{j\omega x}}{\omega-2j} + \lim_{\omega\to 2j}\frac{e^{j\omega x}}{\omega-j}\right) = -e^{-x} + e^{-2x}.$$

(ii) When $x < 0$, we integrate along the semicircular contour on radus $R$ in the *lower* half-plane. But the integrand is analytic everywhere in the lower half-plane (as its only singularities are at $z = j$ and $z = 2j$). Thus $f(x) = 0$ for $x < 0$.

Putting together parts (i) and (ii), we can express it as

$$f(x) = (e^{-2x} - e^{-x})H(x).$$

(b) $F(\omega) = \dfrac{1}{(2-j\omega)^2}$

*Solution.* We may rewrite $F(\omega)$ as

$$F(\omega) = \frac{1}{j^2(-2j-\omega)^2} = -\frac{1}{(\omega+2j)^2},$$

which has a single pole at $\omega = -2j$, which is of order 2, and lies in the lower half plane. Now we integrate

$$f(x) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{j\omega x}}{(\omega+2j)^2}\,d\omega = \lim_{R\to\infty} -\frac{1}{2\pi}\int_{-R}^{+R}\frac{e^{j\omega x}}{(\omega+2j)^2}\,d\omega.$$

Again, there are two cases to consider: $x > 0$ and $x < 0$.

(i) When $x > 0$, we integrate along the semicircular contour on radus $R$ in the upper half-plane, but the integrand is analytic everywhere in the upper half-plane (as its only singularity is at $z = -2j$). Thus $f(x) = 0$ for $x > 0$.

(ii) When $x < 0$, we integrate along the semicircular contour on radus $R$ in the *lower* half-plane. For this integral, we only need the value of the reside computed at $= -2j$. This is computed as

$$\lim_{\omega\to -2j}\frac{d}{d\omega}\left(-e^{j\omega x}\right) = \lim_{\omega\to -2j} -jxe^{j\omega x} = -jxe^{2x}$$

So, for $x < 0$, we have $f(x) = \frac{1}{2\pi}(-2\pi j)(-jxe^{2x}) = -xe^{2x}$.

Putting together parts (i) and (ii), we can express it as

$$f(x) = -xe^{2x}H(-x).$$