ECE 206 – Fall 2019

Quiz 3

October 9, 2018 at 17:30

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University of Waterloo

Notes:

1. Fill in your name (first and last) and student ID number.
2. This quiz contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided on the last page. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
6. You may **not** use your books, notes, calculator, or any other aids on this quiz. The use of personal electronic or communication devices is prohibited.

Problem	Points
1	8
2	4
3	5
4	6
Total:	23

1. Consider a surface $\Sigma = \{\mathbf{\Phi}(u,v) \,|\, u \in (0,1] \text{ and } v \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ determined by a parameterization

$$\mathbf{\Phi}(u,v) = (2u\sin v,\; u\cos v,\; u+1)$$

for all $0 < u \leq 1$ and $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

[2] (a) Determine the normal vector $\boldsymbol{n}_{\mathbf{\Phi}}(u,v)$ to the surface Σ.

Solution: The partial derivatives of the parameterization function with respect to u and v are

$$\frac{\partial \mathbf{\Phi}}{\partial u} = (2\sin v,\; \cos v, 1) \qquad \text{and} \qquad \frac{\partial \mathbf{\Phi}}{\partial v} = (2u\cos v,\; -u\sin v, 0).$$

A normal vector at any point $\mathbf{\Phi}(u,v)$ is therefore

$$\boldsymbol{n}_{\mathbf{\Phi}}(u,v) = \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} = (u\sin(v),\; 2u\cos(v),\; -2u)\,.$$

[3] (b) Determine an implicit representation for the tangent plane at the point $\boldsymbol{r}_0 = (0,1,2)$ on the surface Σ by describing the plane in terms of an equation of the form $\_x + \_y + \_z = \_$.

Solution: The point $\boldsymbol{r}_0$ is parameterized as $\boldsymbol{r}_0 = \mathbf{\Phi}(1,0)$,with $u = 1$ and $v = 0$. The normal vector at this point is $\boldsymbol{n} = \boldsymbol{n}_{\mathbf{\Phi}}(1,0) = (0,2,-2)$. An equation for the plane tangent to the surface at this point is therefore

$$\big((x,y,z) - (0,1,2)\big) \cdot \boldsymbol{n} = 0.$$

As $(0,1,2) \cdot \boldsymbol{n} = 2 - 4 = -2$ and $(x,y,z) \cdot \boldsymbol{n} = 2y - 2z$, the desired equation for the tangent plane is

$$2y - 2z = 2.$$

We may divide by 2 to obtain the simpler expression $y - z = 1$. An implicit representation for this surface is therefore $\{(x,y,z) : y - z = 1\}$.

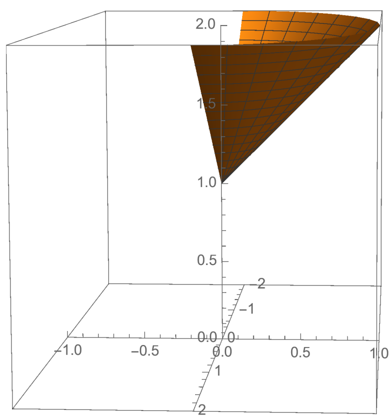
[3] (c) Describe the four grid curves of $\mathbf{\Phi}$ along $u = 1$, and along $v = -\frac{\pi}{2}$, $v = 0$, and $v = \frac{\pi}{2}$. (You may include a sketch as part of your answer.)

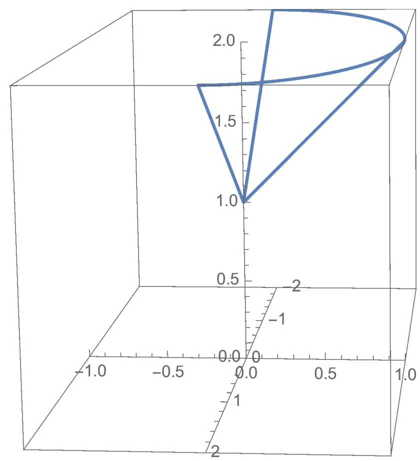
Solution:

- The grid curve defined by $\mathbf{\Phi}(1,v) = (2\sin v, \sin v, 2)$ is the semi-ellipse that is in the half-space $y \geq 0$, is the plane defined by $z = 2$, is centered at the point $(0,0,2)$, and has semi-major axis length 2 in the x-direction and semi-minor axis length 1 in the y-direction.
- The grid curve defined by $\mathbf{\Phi}(u, -\frac{\pi}{2}) = (-2u, 0, u+1)$ is the straight line segment connecting the point $(0,0,1)$ to the point $(-2,0,2)$.

- The grid curve defined by $\mathbf{\Phi}(u, 0) = (0, u, u + 1)$ is the straight line segment connecting the point $(0, 0, 1)$ to the point $(0, 1, 2)$.
- The grid curve defined by $\mathbf{\Phi}(u, \frac{\pi}{2}) = (2u, 0, u + 1)$ is the straight line segment connecting the point $(0, 0, 1)$ to the point $(2, 0, 2)$.

The surface is half of an elliptical cone along the z-axis centered at $(0, 0, 1)$. The resulting surface and the grid curves are displayed below.





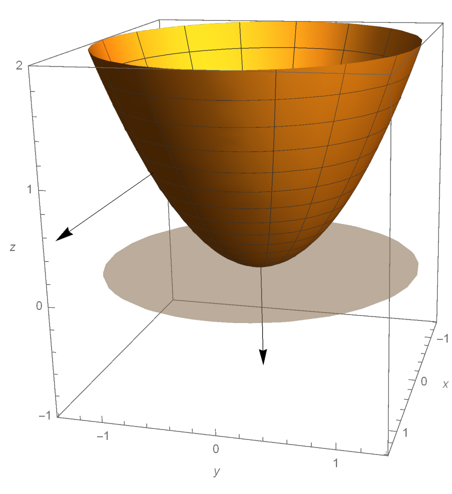
[4] 2. Evaluate the double integral $\iint_D y^2\, dA$ where $D \subseteq \mathbb{R}^2$ is the region defined by $1 \leq x^2 + y^2 \leq 4$.

Solution: Using polar coordinates, this region can be described by $1 \leq r \leq 2$, where we use the transformation $x = r\cos\theta$ and $y = r\sin\theta$. The integral is therefore

$$\begin{aligned}
\iint_D y^2\, dA &= \int_0^{2\pi}\int_1^2 (r^2\sin^2\theta)\,(r)\,dr\,d\theta \\
&= \left(\int_1^2 r^3\,dr\right)\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right) \\
&= \frac{r^4}{4}\bigg|_{r=1}^2 \left(\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta\right) \\
&= \left(4-\frac{1}{4}\right)\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_{\theta=0}^{2\pi} \\
&= \frac{15}{4}(\pi - 0) = \frac{15\pi}{4}.
\end{aligned}$$

[5] 3. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by $\boldsymbol{F}(x,y,z) = (x, y, xyz)$ and let Σ be the surface given by $z = x^2 + y^2$ with $z \in [0,4]$. Determine the flux of $\boldsymbol{F}$ downward through the surface Σ.

Solution: The surface can be visualized as in the following diagram. The (downward) normal vector is depicted at some points.



We first parameterize the surface by the parameterization

$$\boldsymbol{\Phi}(u,v) = (u\cos v,\ u\sin v,\ u^2)$$

for $u \in [0,2]$ and $v \in [0,2\pi]$. We have that

$$\frac{\partial\boldsymbol{\Phi}}{\partial u} = (\cos v, \sin v, 0) \quad\text{and}\quad \frac{\partial\boldsymbol{\Phi}}{\partial v} = (-u\sin v, u\cos v, 2u)$$

so that the normal vector at the point $\boldsymbol{\Phi}(u,v)$ is given by

$$\boldsymbol{n}_{\boldsymbol{\Phi}} = \frac{\partial\boldsymbol{\Phi}}{\partial u}\times\frac{\partial\boldsymbol{\Phi}}{\partial v} = (-2u^2\cos v, -2u^2\sin v, u).$$

However, this vector points *upward*, since its z-component is positive, so we must take the *negative* of this vector as our normal to get the downward flux.

The field at each point on the surface is $\boldsymbol{F}(\boldsymbol{\Phi}(u,v)) = (u\cos v, u\sin v, u^4\cos v\sin v)$. Moreover, we have that

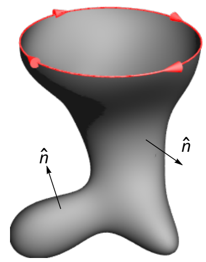
$$\begin{aligned}\boldsymbol{F}(\boldsymbol{\Phi}(u,v))\cdot\boldsymbol{n}_{\boldsymbol{\Phi}}(u,v) &= (u\cos v, u\sin v, u^4\cos v\sin v)\cdot(2u^2\cos v, 2u^2\sin v, -u)\\ &= 2u^3(\cos^2 v+\sin^2 v) - u^5\cos v\sin v\\ &= 2u^3 - u^5\cos v\sin v.\end{aligned}$$

The desired flux can therefore be computed as

$$\begin{aligned}\iint_{\Sigma}\boldsymbol{F}\cdot d\boldsymbol{A} = \iint_{\Sigma}\boldsymbol{F}\cdot d\boldsymbol{A} &= \iint_{D}\boldsymbol{F}(\boldsymbol{\Phi}(u,v))\cdot\boldsymbol{n}_{\boldsymbol{\Phi}}(u,v)\,du\,dv\\ &= \int_0^{2\pi}\int_0^{2}(2u^3 - u^5\cos v\sin v)\,du\,dv\\ &= (2\pi)\,2\,\frac{u^4}{4}\bigg|_{u=0}^{2} = 2^4\pi = 16\pi,\end{aligned}$$

where we note that $\displaystyle\int_0^{2\pi}\cos v\sin v\,dv = 0$.

4. Consider the surface Σ (oriented outwards) shown below. The boundary of Σ is the circle $x^2 + y^2 = 1$ in the xy-plane (i.e., $z = 0$) oriented clockwise when viewed from above.



Σ

Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector fields defined by

$$\boldsymbol{F}(x, y, z) = (yz, xz + x, z), \qquad \text{and} \qquad \boldsymbol{G}(x, y, z) = (ye^{xy}, xe^{xy}, 0).$$

For each part, circle the best answer. Show your work to receive full credit. (Hint: Make use of an important theorem and find a simpler surface with the same boundary.)

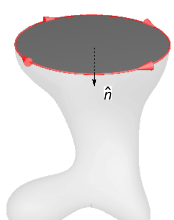
[3] (a) Evaluate the integral $\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA$.

-2π	$-\pi$	0	π	2π

Solution: Use Stokes' theorem to reduce the integral to a line integral over the boundary

$$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = \oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

However, this integral can be further simplified. Consider now the surface S, which is the unit disc in the xy-plane and is the 'top' of the region contained inside Σ, having the same boundary as Σ. This surface (with clockwise boundary and normal oriented downward) is visualized below:



Since Σ and S share a boundary (i.e., $\partial\Sigma = \partial S$), the desired integral becomes

$$\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = \oint_{\partial\Sigma} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA.$$

The (downward) normal vector for S is $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}}$ and the curl of $\boldsymbol{F}$ is $\nabla \times \boldsymbol{F} = (-x, y, 1)$. Thus

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA = -\iint_S (-x, y, 1) \cdot \hat{\boldsymbol{k}}\, dA. = -\iint_S (1)\, dA = -\operatorname{area}(S) = -\pi,$$

since the area of S is π.

[3] (b) Evaluate the integral $\oint_{\partial\Sigma} \boldsymbol{G} \cdot d\boldsymbol{r}$.

-2π	$-\pi$	**[0]**	π	2π

Solution: Note that $\nabla \times \boldsymbol{G} = \mathbf{0}$, so we may use Stokes' Theorem to find that

$$\oint_{\partial\Sigma} \boldsymbol{G} \cdot d\boldsymbol{r} = \iint_{\Sigma} (\nabla \times \boldsymbol{G}) \cdot \hat{\boldsymbol{n}}\, dA = \iint_{\Sigma} 0\, dA = 0.$$

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Change of variable formula

If $\mathbf{\Phi}(u,v) = (x(u,v), y(u,v))$ is a one-to-one C^1-transformation $\mathbf{\Phi} : D \to \mathbb{R}^2$ then

$$\iint_{\mathbf{\Phi}(D)} f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

where the region $\mathbf{\Phi}(D) = \{\mathbf{\Phi}(u,v) \,|\, (u,v) \in D\}$ is the region mapped to. The Jacobian is defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Polar coordinates

For the coordinate transformation defined by $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

Vector calculus identities

$$\nabla \times (f\boldsymbol{F}) = (\nabla f) \times \boldsymbol{F} + f(\nabla \times \boldsymbol{F})$$
$$\nabla \cdot (f\boldsymbol{F}) = (\nabla f) \cdot \boldsymbol{F} + f(\nabla \cdot \boldsymbol{F})$$
$$\nabla \times (\nabla f) = \mathbf{0}$$
$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$$
$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

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(If you want something here marked, be sure to clearly indicate on the question page.)