ECE 206 – Fall 2019

Quiz 4

November 6, 2018 at 17:30

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Solutions

Notes:

1. Fill in your name (first and last) and student ID number.
2. This quiz contains 6 pages (including this cover page) and 3 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided on the last page. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
6. You may **not** use your books, notes, calculator, or any other aids on this quiz. The use of personal electronic or communication devices is prohibited.

Problem	Points
1	5
2	9
3	6
Total:	20

Note: Maxwell's Equations can be found on the formula sheet on the last page for reference.

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[5] 1. Use Maxwell's equations along with a vector calculus identity for ∇ to show that in the case of no currents or charges, $\boldsymbol{B}$ satifies the homogeneous wave equation

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{B} = \boldsymbol{0} \qquad \text{where} \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$

Use Maxwell's equation in source-free case

$$\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}.$$

Taking the curl of both sides yields

$$\underbrace{\nabla \times (\nabla \times \vec{B})}_{\text{vector calculus identity}} = \varepsilon_0 \mu_0 \nabla \times \left(\frac{\partial \vec{E}}{\partial t} \right)$$

we can switch order of derivatives assuming $\vec{E}$ is C^2

$$\Rightarrow \nabla (\underbrace{\nabla \cdot \vec{B}}_{=0}) - \nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} (\underbrace{\nabla \times \vec{E}}_{= -\frac{\partial \vec{B}}{\partial t}})$$

$$\Rightarrow -\nabla^2 \vec{B} = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \vec{B}}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \nabla^2 \vec{B} = \vec{0}$$

$$\Rightarrow \frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = 0 \qquad \text{where } c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

2. Suppose the magnetic vector potential is given by $\boldsymbol{A}(x,y,z,t) = ay\,\hat{\imath} - bt^2x\,\hat{\jmath}$ for some constants a and b.

[3] (a) Determine the magnetic field from this magnetic vector potential.

[6] (b) Let Γ be the curve of the unit circle $x^2+y^2=1$ on the xy-plane oriented counterclockwise when viewed down the z-axis. Determine the value of $\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{r}$ as a function of time. (Hint: Use a theorem from vector calculus, Maxwell's equations, and the $\boldsymbol{B}$-field from part (a).)

a) $\vec{B} = \nabla \times \vec{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & -bt^2x & 0 \end{vmatrix} = \hat{k}\left(-bt^2 - a\right) = -\hat{k}\,(a+bt^2)$$

b) Let Σ be the unit disc in the xy-plane such that $\partial\Sigma = \Gamma$.

Then

$$\oint_\Gamma \vec{E}\cdot d\vec{r} = \iint_\Sigma (\nabla\times\vec{E})\cdot\hat{n}\,dA \qquad \text{(by Stokes')}$$

$$= \iint_\Sigma -\frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,dA \qquad \text{(by Maxwell)}$$

$$= \iint_\Sigma -\frac{\partial}{\partial t}\left(-\hat{k}\,(a+bt^2)\right)\cdot\hat{k}\,dA \qquad \left(\text{since } \hat{n}=\hat{k} \text{ on } \Sigma\right)$$

$$= \iint_\Sigma 2bt\,dA$$

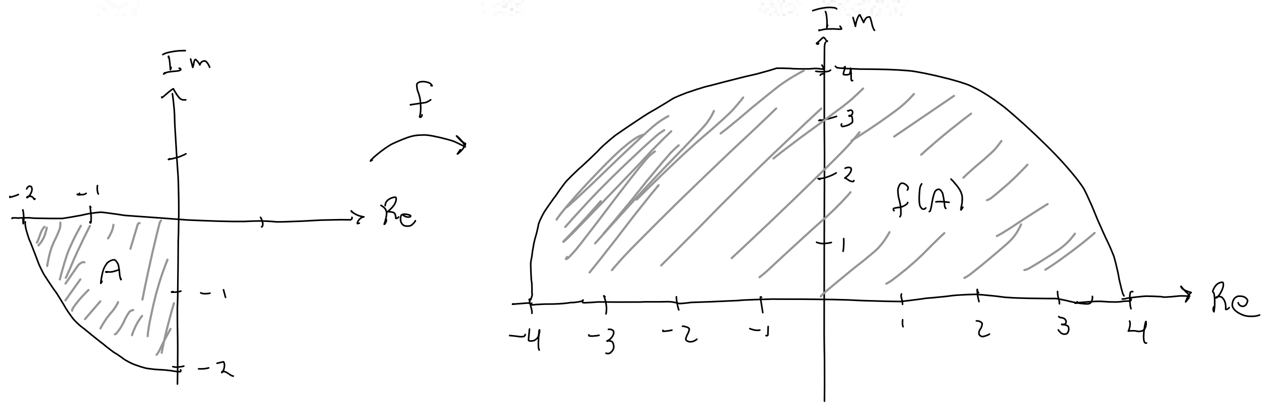
$$= 2bt\ \text{area}(\Sigma)$$

$$= 2\pi bt.$$

[6] 3. Consider the set $A \subset \mathbb{C}$ of complex numbers defined by

$$A = \left\{ z \in \mathbb{C} \,\middle|\, |z| < 2 \text{ and } -\pi < \mathrm{Arg}(z) < -\frac{\pi}{2} \right\}.$$

Sketch A, then find and sketch the image of A under the mapping $f(z) = z^2$.



Quarter of the disc of radius 2 centered at origin in quadrant III

Half of disc of radius 4 centered at the origin in the upper half-plane.

This space is for sketch work or overflow

(If you want something here marked, be sure to clearly indicate on the question page.)

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Vector calculus theorems

Suppose $\boldsymbol{F}$ is a C^1 vector field on $\mathbb{R}^3$.

Stokes' Theorem	$\oint_{\partial\Sigma} \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_{\Sigma} (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{S}$	for all surfaces $\Sigma\subset\mathbb{R}^3$
Divergence Theorem	$\oiint_{\partial\Omega} \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_{\Omega} (\nabla\cdot\boldsymbol{F})\,dV$	for all regions $\Omega\subset\mathbb{R}^3$

Vector calculus identities

$$\nabla\times(f\boldsymbol{F}) = (\nabla f)\times\boldsymbol{F} + f(\nabla\times\boldsymbol{F})$$
$$\nabla\cdot(f\boldsymbol{F}) = (\nabla f)\cdot\boldsymbol{F} + f(\nabla\cdot\boldsymbol{F})$$
$$\nabla\times(\nabla f) = \boldsymbol{0}$$
$$\nabla\cdot(\nabla\times\boldsymbol{F}) = 0$$
$$\nabla\times(\nabla\times\boldsymbol{F}) = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

Maxwell's equations

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{B} = \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}$$