ECE 206 – Fall 2019

Quiz 6

November 27, 2018 at 17:30

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Notes:

1. Fill in your name (first and last) and student ID number.
2. This quiz contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided on the last page. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
6. You may **not** use your books, notes, calculator, or any other aids on this quiz. The use of personal electronic or communication devices is prohibited.

Problem	Points
1	6
2	4
3	4
4	6
Total:	20

1. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $u(x, y) = x^2 + 2y - y^2 + 1$.

[1] (a) Show that u is harmonic.

[4] (b) Find a mapping $f : \mathbb{C} \to \mathbb{C}$ that is differentiable everywhere with $\text{Re}(f(x + jy)) = u(x, y)$ and satisfies $f(1) = 2$. Then find an expression for $f(z)$ only in terms of z (not x or y).

[1] (c) For the mapping f you found in part (b), compute the value of $f'(1)$.

[4] 2. Let $f(z) = \sqrt{z}$ be the principal value of the square root on the region D that is the slit plane

$$D = \left\{ re^{j\theta} \,\middle|\, 0 < r \text{ and } -\pi < \theta < \pi \right\}.$$

Use the Cauchy-Riemann equations in polar form to show that f is differentiable on D and find the derivative $f'(z)$ in terms of z.

[4] 3. Let Γ be the radius 2 centered at the origin (i.e. $\Gamma = \{z \in \mathbb{C} \mid |z| = 2\}$) oriented counterclockwise. Evaluate the integral $\oint_\Gamma z \operatorname{Log}(z)\, dz$ by parameterizing Γ and integrating by parts.

[6] 4. Let Γ be the circle of radius 2 centred at $z = j$ oriented counterclockwise. Evaluate the following integrals. Be sure to explain your reasoning for full credit.

(a) $\displaystyle\oint_\Gamma \frac{\cos z}{z^2(z^2+4)}\,dz$

(b) $\displaystyle\oint_\Gamma \frac{\sinh z}{(z-10)^3}\,dz$

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Complex trigonometric and hyperbolic identities For all complex numbers $z \in \mathbb{C}$:

$$\cos z = \frac{e^{jz}+e^{-jz}}{2} \qquad \sin z = \frac{e^{jz}-e^{-jz}}{2j} \qquad \cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$\cos jz = \cosh z \qquad \sin jz = j\sinh z$$

Cauchy-Riemann equations .
In Cartesian form

$$u_x = v_y \qquad u_y = -v_x$$

and if f is differentiable the derivative is given by $f'(x+jy) = u_x(x,y) + jv_x(x,y)$.
In polar form

$$u_r = \frac{1}{r}v_\theta \qquad v_r = -\frac{1}{r}u_\theta \quad .$$

and if f is differentiable the derivative is given by $f'(re^{j\theta}) = e^{-j\theta}\big(u_r(r,\theta) + jv_r(r,\theta)\big)$.

Generalized Cauchy integral formula For any integer n,

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$

where Γ is any simple closed contour going counterclockwise around the point z_0 in a simply connected region D where f is differentiable.