MATH 135 — Fall 2021
Practice Problems (Solutions)– Chapter 4

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Note. The *floor function* takes a real number x as input and outputs the greatest integer $\lfloor x \rfloor$ that is less than or equal to x. For example,

$$\lfloor 1.2 \rfloor = 1, \qquad \lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -1.3 \rfloor = -2, \qquad \text{and} \qquad \left\lfloor \frac{1}{2} \right\rfloor = 0.$$

For most of the following problems, use induction unless otherwise stated.

1. Prove for all numbers $n \in \mathbb{N}$ that

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = 2^{n-1}.$$

(Note: Induction will not be helpful here. Try out a few small values of n to see if you find a pattern and use Binomial Theorem instead.)

Solution. First, let's try expanding out the sum in question for a few small values of n to see what it looks like. For each $n \in \mathbb{N}$, define $s(n)$ to be the sum in question.

- When $n = 1$, we have $\lfloor 1/2 \rfloor = 0$ and

$$s(1) = \sum_{j=0}^{0} \binom{1}{2j} = \binom{1}{0} = 1 = 2^0$$

.

- When $n = 4$, we have $\lfloor 4/2 \rfloor = 2$ and

$$s(4) = \sum_{j=0}^{2} \binom{4}{2j} = \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 1 + 6 + 1 = 8 = 2^3$$

.

- When $n = 7$, we have $\lfloor 7/2 \rfloor = 3$ and

$$s(7) = \sum_{j=0}^{3} \binom{7}{2j} = \binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 1 + 21 + 35 + 7 = 64 = 2^6$$

.

We see that $s(n)$ is the sum of all $\binom{n}{m}$ where m is an even integer. Another way to write this sum is as

$$s(n) = \sum_{\substack{m=0 \\ m \text{ is even}}}^{n} \binom{n}{m} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor n/2 \rfloor}$$

where we sum only over the even integers between 0 and n. Now, let's explore what happens when we use the Binomial Theorem to expand the sums of $(1+1)^n$ and $(1-1)^n$. For example, note that

$$2^4 = (1+1)^4 = \sum_{m=0}^{4} \binom{n}{4} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

and

$$0 = (1-1)^4 = \sum_{m=0}^{4} \binom{n}{4} (-1)^m = \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}.$$

If we add these two equalities together, we find

$$2^4 + 0 = 2\binom{4}{0} + 2\binom{4}{2} + 2\binom{4}{4}$$

(where the binomial coefficients with odd lower indices are cancelled out). Dividing by two yields the result

$$2^3 = \binom{4}{0} + \binom{4}{2} + \binom{4}{4}.$$

To prove the claim, we can follow this pattern for arbitrary n.

Proof. Let $n \in \mathbb{N}$ be arbitrary. By the Binomial Theorem, we have that

$$2^n = (1+1)^n = \sum_{m=0}^{n} \binom{n}{m}$$

where we can split the summands of the sum on the right into even and odd indices as

$$\begin{aligned} 2^n &= \sum_{\substack{m=0 \\ m \text{ is even}}}^{n} \binom{n}{m} + \sum_{\substack{m=0 \\ m \text{ is odd}}}^{n} \binom{n}{m} \\ &= s(n) + t(n) \end{aligned} \tag{$*$}$$

where we define

$$s(n) = \sum_{\substack{m=0 \\ m \text{ is even}}}^{n} \binom{n}{m} \qquad \text{and} \qquad t(n) = \sum_{\substack{m=0 \\ m \text{ is odd}}}^{n} \binom{n}{m}.$$

Similarly, by the Binomial Theorem, we have

$$0 = 0^n = (1 + (-1))^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m.$$

Because $(-1)^m = 1$ when m is even and $(-1)^m = -1$ when m is odd, this becomes

$$\begin{aligned} 0 &= \binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \cdots + (-1)^m \binom{m}{m} \\ &= \sum_{\substack{m=0 \\ m \text{ is even}}}^{n} \binom{n}{m} - \sum_{\substack{m=0 \\ m \text{ is odd}}}^{n} \binom{n}{m} \qquad (**) \\ &= s(n) - t(n). \end{aligned}$$

Adding together the equalities in $(*)$ and $(**)$ yields

$$2^n = 2s(n)$$

and dividing by 2 yields the desired result. □

2. Prove for all $n \in \mathbb{N}$ that

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

Solution.

Proof. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$. We will prove that $P(n)$ holds for all $n \in \mathbb{N}$ by induction.

- Base case: When $n = 1$, we have

$$\sum_{j=1}^{1} \frac{1}{j(j+1)} = \frac{1}{2} = \frac{1}{1+1}$$

and thus $P(1)$ is true.

- Induction step: Let $k \in \mathbb{N}$ be arbitrary and suppose that $P(k)$ is true. That is, suppose that

$$\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}. \qquad \text{(Induction Hypothesis)}$$

Now

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j(j+1)} &= \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{by IH} \\
&= \frac{1}{k+1}\left(k + \frac{1}{k+2}\right) \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

and thus $P(k+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. □

3. Prove for all natural numbers $n \geq 2$ that

$$\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

Solution.

Proof. For each $n \in \mathbb{N}$ let $P(n)$ be the statement that $\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. We will prove that $P(n)$ is true for all $n \geq 2$ by induction.

- Base case: Note that $\sqrt{2} < 2$ and thus

$$\sqrt{2} = \frac{\sqrt{2}+\sqrt{2}}{2} < \frac{2+\sqrt{2}}{2} = 1 + \frac{1}{\sqrt{2}} = \sum_{k=1}^{2} \frac{1}{\sqrt{k}},$$

so $P(2)$ is true.

- Induction step: Let $m \in \mathbb{N}$ be arbitrary such that $m \geq 2$ and suppose that $P(m)$ is true. That is, suppose that

$$\sqrt{m} < \sum_{k=1}^{m} \frac{1}{\sqrt{k}} \tag{IH}$$

Now, by the induction hypothesis,

$$\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{m} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}.$$

It remains to prove that

$$\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}.$$

Now, $\sqrt{m+1} > \sqrt{m}$ and thus

$$\sqrt{m+1}\sqrt{m} > m,$$

as $m > 0$. Adding 1 to both sides yields

$$\sqrt{m+1}\sqrt{m} + 1 > m + 1$$

and dividing both sides by $\sqrt{m+1}$ yields

$$\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}.$$

It follows that $P(m+1)$ holds.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all natural numbers $n \geq 2$. □

4. Consider a sequence defined by $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2 + a_n}$$

for all $n \in \mathbb{N}$. Prove that $\sqrt{2} \leq a_n < 2$ for all $n \in \mathbb{N}$

Solution.

Proof. Let $P(n)$ be the statement that $\sqrt{2} \leq a_n < 2$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

- Base case: Note that $a_1 = \sqrt{2}$ and $\sqrt{2} \leq \sqrt{2} < 2$, so $P(1)$ is true.
- Induction step: Let $k \in \mathbb{N}$ be arbitrary and suppose that $P(k)$ is true. That is, suppose that

$$\sqrt{2} \leq a_k < 2. \tag{IH}$$

Now $a_{k+1} = \sqrt{2 + a_k}$. It follows from the induction hypothesis that

$$\sqrt{2 + \sqrt{2}} \leq \sqrt{2 + a_k} < \sqrt{2 + 2}$$

and thus

$$\sqrt{2} < \sqrt{2 + \sqrt{2}} \leq a_{k+1} < \sqrt{4} = 2,$$

which proves that $P(k+1)$ holds.

It follows from the Principle of Mathematical Induction that $\sqrt{2} \leq a_n < 2$ holds for every $n \in \mathbb{N}$. □

5. Let $r \in \mathbb{R}$ be a real number such that $r + \frac{1}{r}$ is an integer. Prove that $r^n + \frac{1}{r^n}$ is an integer for all $n \in \mathbb{N}$.

Solution.

Proof. Let $r \in \mathbb{R}$ and suppose that $r + \frac{1}{r}$ is an integer. For each $n \in \mathbb{N}$ let $P(r, n)$ be the statement "$r^n + \frac{1}{r^n}$ is an integer". We will prove that $P(r, n)$ is true for all $n \in \mathbb{N}$ by strong induction.

- Base case: Note that $r^1 + \frac{1}{r^1} = r + \frac{1}{r}$, which is an integer by assumption. Thus $P(r, 1)$ is true.
- Induction step: Let $k \in \mathbb{N}$ be arbitrary and suppose that $P(r, 1)$, $P(r, 2)$, ..., and $P(r, k)$ are all true. That is, suppose that

$$r^m + \frac{1}{r^m} \text{ is an integer for each } m \in \{1, 2, \ldots, k\}. \tag{IH}$$

Now,

$$\left(r^k + \frac{1}{r^k}\right)\left(r + \frac{1}{r}\right) = r \cdot r^k + r\frac{1}{r^k} + \frac{1}{r}r^k + \frac{1}{r}\frac{1}{r^k}$$
$$= r^{k+1} + \frac{1}{r^{k+1}} + r^{k-1} + \frac{1}{r^{k-1}}.$$

It follows that

$$r^{k+1} + \frac{1}{r^{k+1}} = \left(r^k + \frac{1}{r^k}\right)\left(r + \frac{1}{r}\right) - \left(r^{k-1} + \frac{1}{r^{k-1}}\right),$$

where $r^k + \frac{1}{r^k}$, $r^{k-1} + \frac{1}{r^{k-1}}$, and $r + \frac{1}{r}$ are integers by the induction hypothesis. It follows that $r^{k+1} + \frac{1}{r^{k+1}}$ is an integer and thus $P(r, k+1)$ is true.

By the Principle of Strong Induction, it follows that $P(r, n)$ is true for every $n \in \mathbb{N}$. □

6. Consider a sequence $y_1, y_2, \ldots$ defined by $y_1 = 1$ and

$$y_n = 2 \cdot y_{\lfloor \frac{n}{2} \rfloor}$$

for all $n \geq 2$. Prove that $y_n \leq n$ for every $n \in \mathbb{N}$.

Solution.

Proof. Let $P(n)$ be the statement that $y_n \leq n$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction.

- Base case: Note that $y_1 = 1$ and thus $y_1 \leq 1$, so $P(1)$ is true.
- Induction step: Let $k \in \mathbb{N}$ be arbitrary. Suppose that, for every $m \in \{1, 2, \ldots k\}$, $P(m)$ is true. That is, suppose that $y_m \leq m$ holds whenever $1 \leq m \leq k$ (IH).

Note that $k \geq 1$ and thus $k+1 \geq 2$, so $\frac{k+1}{2} \geq 1$. Hence

$$1 \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq \frac{k+1}{2} < k+1$$

and thus $\lfloor \frac{k+1}{2} \rfloor$ is an integer between 1 and k. It follows from the induction hypothesis that

$$y_{\lfloor \frac{k+1}{2} \rfloor} \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq \frac{k+1}{2}.$$

Now

$$y_{k+1} = 2y_{\lfloor \frac{k+1}{2} \rfloor} \leq 2\frac{k+1}{2} = k+1$$

and thus $y_{k+1} \leq k+1$, which proves that $P(k+1)$ holds.

It follows from the Principle of Strong Induction that $y_n \leq n$ for every $n \in \mathbb{N}$. ☐

7. The *Fibonacci sequence* $f_1, f_2, \ldots$ is defined by $f_1 = 1$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2}$$

for all $n \geq 3$. Prove the following facts about the Fibonacci sequence.

(a) For all $n \geq 2$, it holds that $f_n < \left(\frac{7}{4}\right)^{n-1}$.

Solution.

Proof. We will prove that $f_n < \left(\frac{7}{4}\right)^{n-1}$ holds for all $n \in \mathbb{N}$ by strong induction.

- Base case: Note that $f_2 = 1 < \frac{7}{4} = \left(\frac{7}{4}\right)^1$. Also,

$$f_3 = 2 = \frac{32}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^2.$$

  Thus we have shown that $f_n \leq (\frac{7}{4})^{n-1}$ holds when $n = 2$ and when $n = 3$.

- Induction step: Let k be an integer such that $k \geq 3$ and suppose that $f_m \leq (\frac{7}{4})^m$ holds for every $m \in \{1, 2, \ldots k\}$ (IH). By the induction hypothesis, it holds that

$$f_k < \left(\frac{7}{4}\right)^k \qquad \text{and} \qquad f_{k-1} < \left(\frac{7}{4}\right)^{k-1}.$$

Now,

$$\begin{aligned} f_{k+1} &= f_k + f_{k+1} \\ &< \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2} \\ &= \left(\frac{7}{4}\right)^{k-2} \left(\frac{7}{4} + 1\right) \\ &= \left(\frac{7}{4}\right)^{k-2} \frac{11}{4} \\ &= \left(\frac{7}{4}\right)^{k-2} \frac{44}{16} \\ &< \left(\frac{7}{4}\right)^{k-2} \frac{49}{16} \\ &= \left(\frac{7}{4}\right)^{k-2} \left(\frac{7}{4}\right)^{2} \end{aligned}$$

and thus $f_{k+1} < \left(\frac{7}{4}\right)^k$.

It follows from the Principle of Strong Induction that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all $n \geq 2$. □

(b) For all $n \in \mathbb{N}$, it holds that $\sum_{j=1}^{n} f_j = f_{n+2} - 1$.

Solution.

Proof. We proceed by induction.

- Base case: Note that $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$. Hence

$$\sum_{j=1}^{1} f_j = f_1 = 1 = 2 - 1 = f_3 - 1$$

and

$$\sum_{j=1}^{2} f_j = f_1 + f_2 = 1 + 1 = 3 - 1 = f_4 - 1.$$

- Induction step: Let k be an integer such that $k \geq 2$ and suppose that

$$\sum_{j=1}^{k} f_j = f_{k+2} - 1. \tag{IH}$$

Now $f_{k+3} = f_{k+1} + f_{k+2}$ by the definition of the Fibonacci sequence, thus

$$\begin{aligned}\sum_{j=1}^{k+1} f_j &= \sum_{j=1}^{k} f_j + f_{k+1} \\ &= (f_{k+2} - 1) + f_{k+1} && \text{by IH} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1.\end{aligned}$$

Hence $\sum_{j=1}^{k+1} f_j = f_{(k+1)+2} - 1$.

It follows from the Principle of Mathematical Induction that $\sum_{j=1}^{n} f_j = f_{n+2} - 1$ holds for all $n \in \mathbb{N}$. □

(c) For all $n \in \mathbb{N}$, it holds that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$.

Solution.

Proof. We proceed by induction.

- Base case: Note that $f_1 = 1$, $f_2 = 1$, and $f_3 = 2$. Hence

$$\sum_{j=1}^{1} f_j^2 = f_1^2 = 1^2 = 1 \cdot 1 = f_1 f_2$$

and

$$\sum_{j=1}^{2} f_j^2 = f_1^2 + f_2^2 = 1^2 + 1^2 = 2 = 1 \cdot 2 = f_2 f_3.$$

- Induction step: Let k be an integer such that $k \geq 2$ and suppose that

$$\sum_{j=1}^{k} f_j^2 = f_k f_{k+1}. \tag{IH}$$

Note that $f_{k+2} = f_k + f_{k+1}$ by the definition of the Fibonacci sequence and thus

$$\begin{aligned}\sum_{j=1}^{k+1} f_j^2 &= \sum_{j=1}^{k} f_j^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 && \text{by IH} \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}.\end{aligned}$$

Hence $\sum_{j=1}^{k+1} f_j = f_{k+1} f_{(k+1)+1}$.

It follows from the Principle of Mathematical Induction that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$ holds for all $n \in \mathbb{N}$. □

(d) Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. It holds that $f_n = \frac{a^n - b^n}{\sqrt{5}}$ for all $n \in \mathbb{N}$

Solution.

Proof. We proceed by strong induction. First note that

$$a^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + a.$$

Similarly,

$$b^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = 1 + b.$$

Hence $1 + a = a^2$ and $1 + b = b^2$.

- Base case: Note that

$$\frac{a-b}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}}{\sqrt{5}} = 1 = f_1$$

  and using the facts that $1 + a = a^2$ and $1 + b = b^2$ note that

$$\frac{a^2 - b^2}{\sqrt{5}} = \frac{1 + a - (1 + b)}{\sqrt{5}} = \frac{a-b}{\sqrt{5}} = 1 = f_2.$$

- Induction step: Let $k \geq 2$ and suppose that

$$f_m = \frac{a^m - b^m}{\sqrt{5}} \tag{IH}$$

holds for every $m \in \{1, 2, \ldots, k\}$. Now

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &= \frac{a^k - b^k}{\sqrt{5}} + \frac{a^{k-1} - b^{k-1}}{\sqrt{5}} && \text{by IH} \\ &= \frac{a^{k-1}(a+1) - b^{k-1}(b+1)}{\sqrt{5}} \\ &= \frac{a^{k-1}a^2 - b^{k-1}b^2}{\sqrt{5}} && \text{because } 1 + a = a^2 \text{ and } 1 + b = b^2 \\ &= \frac{a^{k+1} - b^{k+1}}{\sqrt{5}}. \end{aligned}$$

It follows from the Principle of Mathematical Induction that $f_n = \frac{a^n - b^n}{\sqrt{5}}$ holds for all $n \in \mathbb{N}$. $\square$