MATH 135 — Fall 2021
Practice Problems – Chapter 4

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Note. The *floor function* takes a real number x as input and outputs the greatest integer $\lfloor x \rfloor$ that is less than or equal to x. For example,

$$\lfloor 1.2 \rfloor = 1, \qquad \lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -1.3 \rfloor = -2, \qquad \text{and} \qquad \left\lfloor \frac{1}{2} \right\rfloor = 0.$$

For most of the following problems, use induction unless otherwise stated.

1. Prove for all numbers $n \in \mathbb{N}$ that
$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = 2^{n-1}.$$
(Note: Induction will not be helpful here. Try out a few small values of n to see if you find a pattern and use Binomial Theorem instead.)

2. Prove for all $n \in \mathbb{N}$ that
$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

3. Prove for all natural numbers $n \geq 2$ that
$$\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

4. Consider a sequence defined by $a_1 = \sqrt{2}$ and
$$a_{n+1} = \sqrt{2 + a_n}$$
for all $n \in \mathbb{N}$. Prove that $\sqrt{2} \leq a_n < 2$ for all $n \in \mathbb{N}$

5. Let $r \in \mathbb{R}$ be a real number such that $r + \frac{1}{r}$ is an integer. Prove that $r^n + \frac{1}{r^n}$ is an integer for all $n \in \mathbb{N}$.

6. Consider a sequence $y_1, y_2, \ldots$ defined by $y_1 = 1$ and

$$y_n = 2 \cdot y_{\lfloor \frac{n}{2} \rfloor}$$

for all $n \geq 2$. Prove that $y_n \leq n$ for every $n \in \mathbb{N}$.

7. The *Fibonacci sequence* $f_1, f_2, \ldots$ is defined by $f_1 = 1$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2}$$

for all $n \geq 3$. Prove the following facts about the Fibonacci sequence.

(a) For all $n \geq 2$, it holds that $f_n < \left(\frac{7}{4}\right)^{n-1}$.

(b) For all $n \in \mathbb{N}$, it holds that $\sum_{j=1}^{n} f_j = f_{n+2} - 1$.

(c) For all $n \in \mathbb{N}$, it holds that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$.

(d) Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. It holds that $f_n = \frac{a^n - b^n}{\sqrt{5}}$ for all $n \in \mathbb{N}$