MATH 135 — Fall 2021
Practice Problems (Solutions)– Chapter 5

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Part I

Determine which of the following statements are true and which are false. Prove the true statements. For the false statements, write the negation and prove that.

1. $\forall A \subseteq \mathbb{Z}, \exists B \subseteq \mathbb{Z}$ so that $1 \in B - A$.

> **Solution.** This statement is false.
> **Negation**: $\exists A \subseteq \mathbb{Z}$ so that $\forall B \subseteq \mathbb{Z}, 1 \notin B - A$.
> *Proof (of negation).* Let $A = \mathbb{Z}$. Let B be an arbitrary subset of $\mathbb{Z}$. Then $B - A = B - \mathbb{Z} = \emptyset$ and $1 \notin \emptyset$. Therefore $1 \notin B - A$. □
> Note: Any other set A that contains 1 will work as a counterexample.

2. $\forall A \subset \mathbb{Z}, \exists B \subseteq \mathbb{Z}$ so that $1 \notin B - A$.

> **Solution.** This statement is true.
>
> *Proof.* Let A be an arbitrary subset of $\mathbb{Z}$ and let $B = \emptyset$. One has that $B - A = \emptyset - A = \emptyset$ and $1 \notin \emptyset$. Therefore $1 \notin B - A$. □
> Note: Any other set B that does not contain 1 will work.

3. For all sets A, B, and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

> **Solution.** This statement is true.
>
> *Proof.* Let A, B, and C be arbitrary sets. Let $x \in (A \cup B) \cap C$. It follows that $x \in A \cup B$ and $x \in C$.
>
> Case 1: Suppose that $x \in A$. It follows that $x \in A \cup (B \cap C)$.
>
> Case 2: Suppose that $x \notin A$. Because $x \in A \cup B$, it must be the case that $x \in B$. Hence $x \in B \cap C$ as x is also in C. We conclude that $x \in A \cup (B \cap C)$.
>
> In either case, $x \in A \cup (B \cap C)$. This completes the proof. □

4. For all sets A, B, and C, $A \cup (B \cap C) \subseteq (A \cup B) \cap C$.

Solution. This statement is false.
Negation: There exists sets A, B, and C so that $A \cup (B \cap C) \not\subseteq (A \cup B) \cap C$.
Proof (of negation). Let $A = \{1,2\}$, $B = \{2\}$ and $C = \{2\}$. One has that

$$B \cap C = \{2\} \qquad \text{and} \qquad A \cup (B \cap C) = \{1,2\},$$

so $1 \in A \cup (B \cap C)$. However, $A \cup B = \{1,2\}$ and $(A \cup B) \cap C\{2\}$, so $1 \notin (A \cup B) \cap C$. □

5. For all sets A, B, and C, if $A \times B = A \times C$ then $B = C$.

Solution. This statement is false.
Negation: There exists sets A, B, and C so that $A \times B = A \times C$ but $B \neq C$.

Proof (of negation). Let $A = \emptyset$, $B = \{1\}$ and $C = \{2\}$. Hence

$$A \times B = \emptyset \qquad \text{and} \qquad A \times C = \emptyset,$$

but $\{1\} \neq \{2\}$ so $B \neq C$. □

6. For all sets A, B, and C, if $A - B \subseteq C$ then $A - C \subseteq B$.

Solution. This statement is true.

Proof. Let A, B, and C be sets. Assume that $A - B \subseteq C$. (We want to show that $A - C \subseteq B$.) Let $x \in A - C$. This means that $x \in A$ and $x \notin C$. We will prove that $x \in B$ by contradiction. Suppose instead that $x \notin B$. Then $x \in A - B$ since $x \in A$ and $x \notin B$. This means that $x \in C$, since $x \in A - B$ and $A - B \subseteq C$. Thus $x \notin C$ and $x \in C$, a contradiction. So the assumption that $x \notin B$ is wrong, and thus $x \in B$. Therefore $A - C \subseteq B$. □

7. For all sets A, B, and C, if $A \cap B \subseteq C$ and $B \cap C \subseteq A$ then $C \cap A \subseteq B$.

Solution. This statement is false.
Negation: There exists sets A, B, and C so that $A \cap B \subseteq C$ and $B \cap C \subseteq A$ but $C \cap A \not\subseteq B$.

Proof (of negation). Let $A = \{1\}$, $B = \{2\}$, and $C = \{1\}$. Then $A \cap B = \emptyset$ and $B \cap C = \emptyset$ and $\emptyset \subseteq C$ and $\emptyset \subseteq A$. Thus $A \cap B \subseteq C$ and $B \cap C \subseteq A$. However, $C \cap A = \{1\}$ and $\{1\} \not\subseteq \{2\}$. Therefore $C \cap A \not\subseteq B$. □

8. For all sets A, B, and C, if $A - (B \cap C) = \emptyset$ then $A - C = \emptyset$.

Solution. This statement is true.

Proof. Let A, B, and C be sets. Suppose that $A - (B \cap C) = \emptyset$. (We want to show that $A - C = \emptyset$.) Assume for the sake of getting a contradiction that $A - C \neq \emptyset$. There exists an element $x \in A - C$. This means that $x \in A$ and $x \notin C$. Then $x \notin B \cap C$ since $x \notin C$. Thus $x \in A$ and $x \notin B \cap C$, which means that $x \in A - (B \cap C)$. But $A - (B \cap C) = \emptyset$, so $x \notin A - (B \cap C)$. This is a contradiction, so the assumption that $A - C \neq \emptyset$ is wrong. Therefore $A - C = \emptyset$. □

9. For all sets A, B, and C, if $A - C = \emptyset$ then $A - (B \cap C) = \emptyset$.

Solution. This statement is false.
Negation: There exists sets A, B, and C so that $A - C = \emptyset$ but $A - (B \cap C) \neq \emptyset$.

Proof (of negation). Let $A = \{1\}$, $B = \{2\}$, and $C = \{1\}$. Then $A - C = \emptyset$ and $B \cap C = \emptyset$, but $A - (B \cap C) = \{1\}$ and $\{1\} \neq \emptyset$. Therefore $A - (B \cap C) \neq \emptyset$. □

Part II

1. Proof De Morgan's Laws for sets. That is, for all sets A and B, it holds that:
   (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$, and
   (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution.

Proof. Let A and B be sets in some universal set $\mathcal{U}$.

(a) Let $x \in \mathcal{U}$ and note that

$$\begin{aligned} x \in \overline{A \cup B} &\iff x \notin A \cup B \\ &\iff \neg(x \in A \cup B) \\ &\iff \neg(x \in A \text{ OR } x \in B) \\ &\iff x \notin A \text{ AND } x \notin B && \text{by De Morgan's laws} \\ &\iff x \in \overline{A} \text{ AND } x \in \overline{B} \\ &\iff x \in \overline{A} \cap \overline{B}, \end{aligned}$$

which proves that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(b) Let $x \in \mathcal{U}$ and note that

$$\begin{aligned}
x \in \overline{A \cap B} &\iff x \notin A \cap B \\
&\iff \neg(x \in A \cap B) \\
&\iff \neg(x \in A \text{ AND } x \in B) \\
&\iff x \notin A \text{ OR } x \notin B && \text{by De Morgan's laws} \\
&\iff x \in \overline{A} \text{ OR } x \in \overline{B} \\
&\iff x \in \overline{A} \cup \overline{B},
\end{aligned}$$

which proves that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

□

2. Suppose A and B are arbitrary subsets of $\mathbb{Z}$ such that $(2,3) \in A \times B$ and $(3,4) \in A \times B$, but that $(1,3) \notin A \times B$.

(a) Find another element in $A \times B$ that is not $(2,3)$ or $(3,4)$. Explain.

Solution. $(2,4) \in A \times B$ and $(3,3) \in A \times B$.

Proof. Note that $(2,3) \in A \times B$ means that $2 \in A$ and $3 \in B$. Similarly, $(3,4) \in A \times B$ means that $3 \in A$ and $4 \in B$. Thus $(2,4) \in A \times B$ and $(3,3) \in A \times B$. □

(b) Find another element that is not in $A \times B$. Explain.

Solution. $(1,7) \notin A \times B$.

Proof. Note that $(1,3) \notin A \times B$ means that $1 \notin A$ or $3 \notin B$. However, we know from part (a) that $3 \in B$, so it must be the case that $1 \notin A$. Thus $(1,7) \notin A \times B$, since $1 \notin A$. □

Note: Any number other than 7 will also work.

3. Suppose A and B are arbitrary subsets of $\mathbb{Z}$ such that $A \cap B = \{1\}$.

(a) Find an element of $A \times B$. Explain why it is an element of $A \times B$.

Solution. $(1,1) \in A \times B$.

Proof. Note that $1 \in A \cap B$ so $1 \in A$ and $1 \in B$. Thus $(1,1) \in A \times B$, by definition of the product of sets. □

(b) Find an element of the complement $\overline{A \times B}$. (Here, assume that the universal set is $\mathbb{Z} \times \mathbb{Z}$.) Explain.

Solution. $(2,2) \notin A \times B$.

Proof. Suppose instead that $(2,2) \in A \times B$. Then $2 \in A$ and $2 \in B$ which means that $2 \in A \cap B$. But $A \cap B = \{1\}$ and $2 \notin \{1\}$, and thus $2 \notin A \cap B$. This is a contradiction so the supposition that $(2,2) \in A \times B$ is wrong. Therefore $(2,2) \notin A \times B$. □

Note: Any number other than 2 will also work.