MATH 135 — Fall 2021
Practice Problems – Chapters 6, 7, and 8

Mark Girard

March 24, 2025

Topics: divisibility, gcd, linear Diophantine equation, Euclidean Algorithm, prime factorizations, and modular arithmetic. (Problems are in no particular order.)

1. Determine $d = \gcd(339, -2145)$ and find integers s and t such that $399s - 2145t = d$.

2. Prove the following statement:

$$\text{For all } a, b, c \in \mathbb{Z}, \text{ if } \gcd(a, b) = 1 \text{ and } a \mid c \text{ and } b \mid c, \text{ then } ab \mid c.$$

3. Prove or disprove the following statement

$$\text{For all integers } x, y, z \in \mathbb{Z}, \text{ if } x \mid yz \text{ then } x \mid y \text{ or } x \mid z.$$

4. Prove, for all positive integers d, m, and n, that if $d = \gcd(m, n)$ then for all positive integers k it holds that $\gcd(m, nk) = \gcd(m, dk)$.

5. Let a and b be integers, let $d = \gcd(a, b)$, and consider the set $S = \{ax + by : x, y \in \mathbb{Z}\}$. Prove that

$$S = \{kd : k \in \mathbb{Z}\}$$

6. Prove that, for all prime numbers p and q, $\{px + qy : x, y \in \mathbb{Z}\} = \mathbb{Z}$ if and only if $p \neq q$.

7. Let $a = 3^2 5^3 7^4 13^1$, $b = 5^1 7^2 13^2 23^9$, and $c = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 23$.

   (a) Determine $\gcd(a, b)$.

   (b) What is the smallest integer t such that $a \mid c^t$ and $b \mid c^t$?

8. Suppose $a \in \mathbb{Z}$ and consider the statement P: "if $24 \mid a^2$ then $36 \mid a^3$".

   (a) Prove P.

   (b) Prove or disprove the converse of P.

9. Suppose a and b are positive integers and let c be an integer such that $\gcd(a, b) \mid c$. Prove that there exists a unique integer solution (x', y') to the linear Diophantine Equation

$$ax + by = c$$

such that $0 \leq x' < \frac{b}{\gcd(a,b)}$.

10. Suppose that Canada Post issued 49¢ and 53¢ stamps. How many different ways could you purchase exactly $100 worth of these kinds of stamps?

11. Let n be a positive integer. Prove the following statements.
    (a) If n is odd, then $n^2 \equiv 1 \pmod 8$.
    (b) If $n^2 \not\equiv 1 \pmod 3$, then $n \equiv 0 \pmod 3$.

12. Solve the equation $[9][x] = [5]$ in $\mathbb{Z}_{43}$.

13. (a) Find the units digit of 6012016^{20} (in base 10).
    (b) Find the last two digits of 7^{1942} in base 10.

14. Prove the following facts about the binomial coefficient.
    (a) For all non-negative integers $n, k \in \mathbb{Z}$, it holds that $\binom{n}{k} \in \mathbb{Z}$.
    (b) Let p be a prime number. It holds that

    $$\binom{p}{k} \equiv 0 \pmod p$$

    for all $k \in \{1, 2, \ldots, p-1\}$.

15. Prove the following statements.
    (a) The sum of any three consecutive natural numbers is divisible by 3.
    (b) The sum of any four consecutive natural numbers is NOT divisible by 4.

16. Let $x \in \mathbb{Z}$. Prove that $4x^2 + x + 3$ is not divisible by 5.

17. Let p be a prime number. Prove the following statement:

$$\text{There exists an integer } n \in \mathbb{Z} \text{ such that } n^3 = p + 8 \iff p = 19.$$

18. Let $a, b \in \mathbb{Z}$ and let p be a prime number. Prove that $(a+b)^p \equiv a^p + b^p \pmod p$.