MATH 135 — Fall 2021
Practice Problems – Chapters 9 and 10

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Topics: Complex numbers and polynomials.

1. Express the following complex numbers in **standard form**. (That is, find $x, y \in \mathbb{R}$ such that $z = x + yi$.)

   (a) $\dfrac{1+i}{1-i}$

   (b) $\dfrac{3+i}{2-5i}$

   (c) $\dfrac{(\sqrt{3}+i)^2}{(\sqrt{3}-i)(1+\sqrt{3}i)}$

   (d) $(i-1)^4$

2. Express the following complex numbers in **polar form**. (That is, find real numbers $r, \theta \in \mathbb{R}$ such that $z = r(\cos\theta + i\sin\theta)$, $0 \leq r$, and $0 \leq \theta < 2\pi$.)

   (a) $\dfrac{1+i}{1-i}$

   (b) $\dfrac{5+i}{2i-3}$

   (c) $(i-\sqrt{3})^7$

3. Identify and sketch the set of points in the complex plane satisfying:

   (a) $|z| = 1$

   (b) $|z - i - 1| \leq 2$

   (c) $|z - 1| < |z|$

4. Prove for all integers $n \in \mathbb{Z}$ that $\operatorname{Re}((\sqrt{3}+i)^n) = 0$ if and only if $n \equiv 3 \pmod 6$

5. Prove that, for all complex numbers $z \in \mathbb{C}$,

$$|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|.$$

6. Let $f(x) = x^3 - 7x^2 + 17x - 15$.

   (a) Show that $f(2+i) = 0$.

   (b) Use the fact that $2+i$ is a root of $f(x)$ to completely factor $f(x)$ in both $\mathbb{C}[x]$ and $\mathbb{R}[x]$.

7. Let $f(x) = x^4 - 3x^3 + 5x^2 - 3x + 4$. Verify that $f(i) = 0$ and use this fact to completely factorize $f(x)$ in $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

8. Let $z, w \in \mathbb{C}$. Prove that $|z + iw| = |z - iw|$ if and only if $z\overline{w} \in \mathbb{R}$.