MATH 135 — Fall 2021
Practice Problems (Solutions)– Chapters 9 and 10

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December 3, 2021

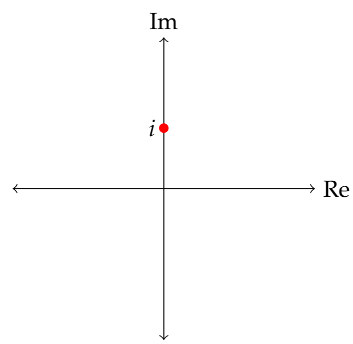
Topics: Complex numbers and polynomials.

1. Express the following complex numbers in **standard form**. (That is, find $x, y \in \mathbb{R}$ such that $z = x + yi$.)

   (a) $\dfrac{1+i}{1-i}$

   Solution. Note that we can multiply the top and bottom by the conjugate of $1-i$ to find

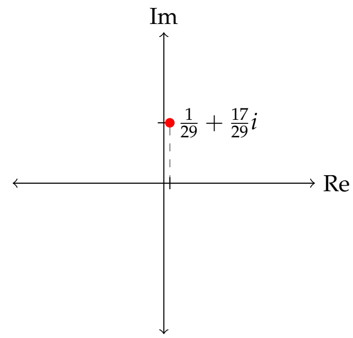
   $$\frac{1+i}{1-i} = \frac{1+i}{1-i}\frac{1+i}{1+i} = \frac{(1+i)^2}{|1+i|^2} = \frac{(1-1)+i(1+1)}{1^2+1^2} = \frac{0+2i}{2} = i.$$

   (b) $\dfrac{3+i}{2-5i}$

   Solution. Note that we can multiply the top and bottom by the conjugate of $2-5i$ to find

   $$\begin{aligned}\frac{3+i}{2-5i} &= \frac{3+i}{2-5i}\frac{2+5i}{2+5i} = \frac{(3+i)(2+5i)}{|2-5i|^2} \\ &= \frac{(3\cdot 2 - 1\cdot 5) + i(3\cdot 5 + 1\cdot 2)}{2^2+5^2} = \frac{(6-5)+i(15+2)}{4+25} = \frac{1}{29} + \frac{17}{29}i.\end{aligned}$$



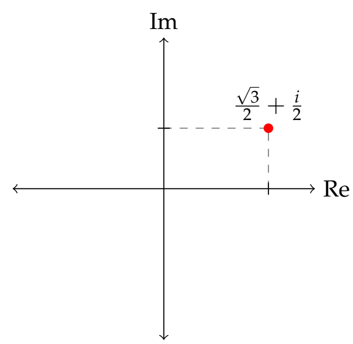
(c) $\dfrac{(\sqrt{3}+i)^2}{(\sqrt{3}-i)(1+\sqrt{3}i)}$

Solution. Note that we can multiply out the denominator to find

$$\begin{aligned}(\sqrt{3}-i)(1+\sqrt{3}i) = (\sqrt{3}+\sqrt{3})+(3-1)i &= 2\sqrt{3}+2i \\ &= 2(\sqrt{3}+i)\end{aligned}$$

and thus

$$\begin{aligned}\frac{(\sqrt{3}+i)^2}{(\sqrt{3}-i)(1+\sqrt{3}i)} &= \frac{(\sqrt{3}+i)^2}{2(\sqrt{3}+i)} \\ &= \frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2}+\frac{1}{2}i.\end{aligned}$$



(d) $(i-1)^4$

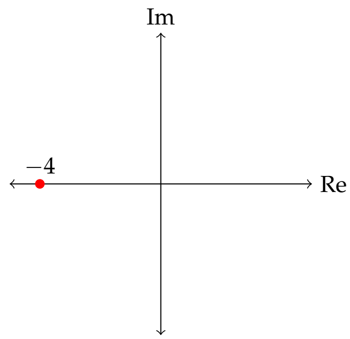
Solution. Note that $i-1$ can be expressed in polar form as

$$i-1 = \sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right),$$

and thus

$$(i-1)^4 = \sqrt{2}^4 \operatorname{cis}\left(\frac{3\pi}{4}\right)^4 = 4\operatorname{cis}(3\pi) = 4\big(\cos(3\pi) + i\sin(3\pi)\big) = 4(-1+0i) = -4,$$

hence $(i-1)^4 = -4$

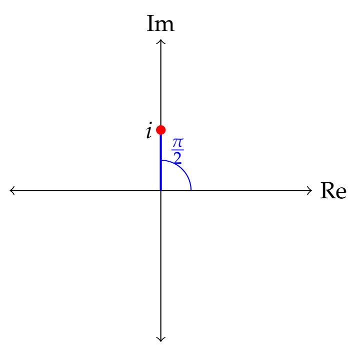


2. Express the following complex numbers in **polar form**. (That is, find real numbers $r, \theta \in \mathbb{R}$ such that $z = r(\cos\theta + i\sin\theta)$, $0 \le r$, and $0 \le \theta < 2\pi$.)

   (a) $\dfrac{1+i}{1-i}$

   Solution. From 1a we have that $\frac{1+i}{1-i} = i$, which in polar form is

   $$i = \operatorname{cis}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right).$$



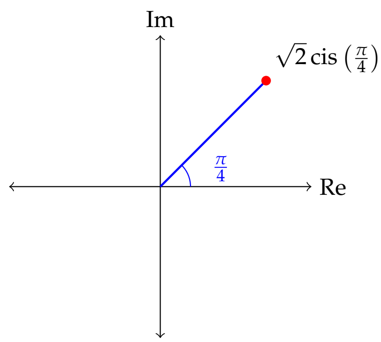
   (b) $\dfrac{5+i}{2i-3}$

Solution. Note that

$$\begin{aligned}
\frac{5+i}{2i-3} = \frac{5+i}{2i-3}\frac{-2i-3}{-2i-3} &= \frac{(5+i)(-2i-3)}{|2i-3|^2} \\
&= \frac{(-15+2)+(-10-3)i}{2^2+3^2} \\
&= \frac{13+13i}{4+9} \\
&= \frac{13(1+i)}{13} = 1+i
\end{aligned}$$

and in polar form this is

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right).$$



(c) $(i-\sqrt{3})^7$

Solution. Note that we can express $i-\sqrt{3}$ in polar form as

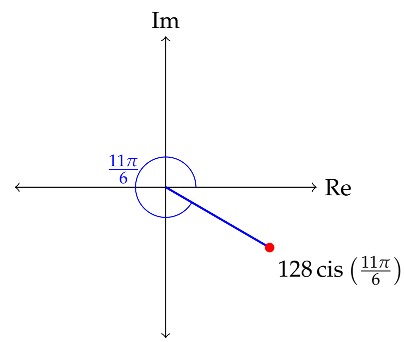
$$\begin{aligned}
i-\sqrt{3} &= 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\
&= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\
&= 2\operatorname{cis}\left(\frac{5\pi}{6}\right).
\end{aligned}$$

Hence, by De Moivre's Theorem,

$$\begin{aligned}(i - \sqrt{3})^7 &= \left(2\,\text{cis}\left(\frac{5\pi}{6}\right)\right)^7 \\ &= 2^7\,\text{cis}\left(7\frac{5\pi}{6}\right) \\ &= 128\,\text{cis}\left(\frac{35\pi}{6}\right) \\ &= 128\,\text{cis}\left(\frac{11\pi}{6} + 4\pi\right) \\ &= 128\,\text{cis}\left(\frac{11\pi}{6}\right) \\ &= 128\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)\end{aligned}$$

Note that in standard form this is

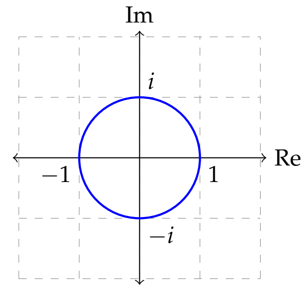
$$128\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right) = 128\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 64\sqrt{3} + 64i.$$



3. Identify and sketch the set of points in the complex plane satisfying:

   (a) $|z| = 1$

Solution. For a complex number in Cartesian coordinates $z = x + yi$, note that $|z| = \sqrt{x^2 + y^2} = 1$. Hence a number satisfied $|z| = 1$ if and only if $x^2 + y^2 = 1$. This equation describes the unit circle in the complex plane that is centered at the origin. This can be visualized as in the following figure.

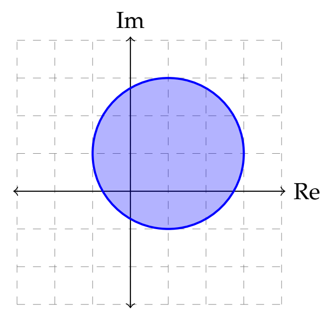


(b) $|z - i - 1| \leq 2$

Solution. Let $z = x + yi$ be the standard form of a complex number. Note that

$$|z - i - 1|^2 = |(x-1) + (y-1)i|^2 = (x-1)^2 + (y-1)^2.$$

Thus z satisfies the inequality $|z - i - 1| \leq 2$ if and only if $(x-1)^2 + (y-1)^2 \leq 4$. The set of points satisfying $(x-1)^2 + (y-1)^2 \leq 4$ form a filled-in disk centered at $(1, 1)$ and having radius 2. This can be visualized as in the following figure.

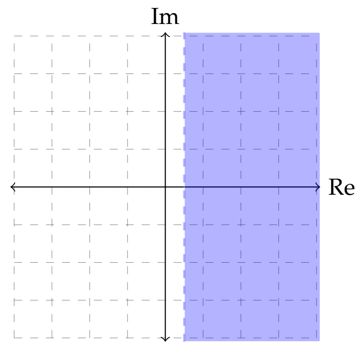


(c) $|z - 1| < |z|$

Solution. This inequality is equivalent to $0 < |z|^2 - |z-1|^2$. For a complex number Cartesian coordinates $z = x + yi$, note that

$$\begin{aligned} |z|^2 - |z-1|^2 &= x^2 + y^2 - \left((x-1)^2 + y^2\right) \\ &= 2x - 1, \end{aligned}$$

and thus z satisfies $0 < |z|^2 - |z-1|^2$ if and only if $0 < 2x - 1$, or equivalently $x > \frac{1}{2}$. Thus, the region satisfying this inequality is the region of all point that lie to the right of the line defined by the equation $x = \frac{1}{2}$. This can be visualized as in the following figure.



4. Prove for all integers $n \in \mathbb{Z}$ that $\operatorname{Re}((\sqrt{3}+i)^n) = 0$ if and only if $n \equiv 3 \pmod 6$

Solution. Note that we can express $\sqrt{3}+i$ in polar form as

$$\sqrt{3}+i = 2\operatorname{cis}\left(\frac{\pi}{6}\right).$$

Let $n \in \mathbb{Z}$ be arbitrary. Using De Moivre's Theorem, we find that

$$\operatorname{Re}((\sqrt{3}+i)^n) = \operatorname{Re}\left(\left(2\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^n\right) = 2^n \operatorname{Re}\left(\operatorname{cis}\left(\frac{n\pi}{6}\right)\right) = 2^n\cos\left(\frac{n\pi}{6}\right).$$

Moreover, recall that $\cos(\theta) = 0$ if and only if $\theta = \frac{\pi}{2} + k\pi$ for some integer $k \in \mathbb{Z}$. Hence,

$$\begin{aligned}
\operatorname{Re}((\sqrt{3}+i)^n) = 0 &\iff 2^n\cos\left(\frac{n\pi}{6}\right) = 0 \\
&\iff \cos\left(\frac{n\pi}{6}\right) = 0 \\
&\iff \exists k \in \mathbb{Z}, \frac{n\pi}{6} = \frac{\pi}{2} + k\pi \\
&\iff \exists k \in \mathbb{Z}, n\pi = 3\pi + 6k\pi \\
&\iff \exists k \in \mathbb{Z}, n = 3 + 6k \\
&\iff n \equiv 3 \pmod 6,
\end{aligned}$$

as desired.

5. Prove that, for all complex numbers $z \in \mathbb{C}$,

$$|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|.$$

Solution.

Proof. Let $z \in \mathbb{C}$ and let $x, y \in \mathbb{R}$ such that $z = x + yi$. Then the real and imaginary parts of z are

$$\operatorname{Re}(z) = x \qquad \text{and} \qquad \operatorname{Im}(z) = y.$$

Now, note that $x^2 = |x|^2$ and $y^2 = |y|^2$, and thus

$$\begin{aligned}(|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 &= (|x| + |y|)^2 \\ &= |x|^2 + 2|x||y| + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 + (|x| - |y|)^2 \quad \text{[because } 0 \leq (|x| - |y|)^2\text{]} \\ &= 2|x|^2 + 2|y|^2 + 2|x||y| - 2|x||y| \\ &= 2(|x|^2 + |y|^2) \\ &= 2(x^2 + y^2) \\ &= 2|z|^2.\end{aligned}$$

Hence we find that $(|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 \leq 2|z|^2$. Taking the square root of both sides yields

$$|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|.$$

□

6. Let $f(x) = x^3 - 7x^2 + 17x - 15$.

(a) Show that $f(2+i) = 0$.

Solution. Using the binomial theorem, we have

$$(2+i)^3 = 2^3 + 3 \cdot 2^2 \cdot i + 3 \cdot 2 \cdot i^2 + i^3 = 8 + 12i - 6 - i = 2 + 11i$$

and

$$(2+i)^2 = 2^2 + 2 \cdot 2 \cdot i + i^2 = 2 + 4i - 1 = 1 + 4i.$$

Hence

$$\begin{aligned}f(2+i) &= (2+i)^3 - 7(2+i)^2 + 17(2+i) - 15 \\ &= (2 + 11i) - 7 \cdot (1 + 4i) + 17(2+i) - 15 \\ &= 2 + 11i - 7 - 28i + 24 + 17i - 15 \\ &= (2 - 7 + 24 - 15) + (11 - 28 + 17)i \\ &= 0 + 0i = 0.\end{aligned}$$

(b) Use the fact that $2+i$ is a root of $f(x)$ to completely factor $f(x)$ in both $\mathbb{C}[x]$ and $\mathbb{R}[x]$.

Solution. Because $2+i$ is a root of $f(x)$ and $f(x) \in \mathbb{R}[x]$, we know that its conjugate $\overline{2+i} = 2 - i$ is also a root of $f(x)$. Hence, the polynomial

$$\big(x - (2+i)\big)\big(x - (2-i)\big) = x^2 - 4x + 5$$

is a factor of $f(x)$. Performing polynomial long division, we find that

$$\begin{array}{r|r} & x \quad -3 \\ \hline x^2-4x+5 & x^3-7x^2+17x-15 \\ & -x^3+4x^2 \quad -5x \quad\quad\; \\ \hline & -3x^2+12x-15 \\ & 3x^2-12x+15 \\ \hline & 0 \end{array}$$

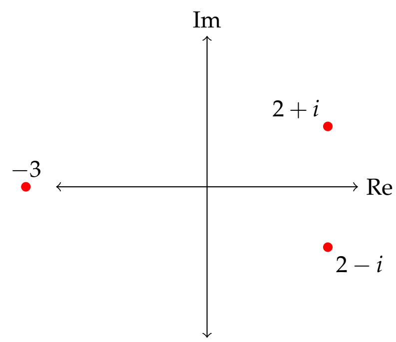
and thus $f(x) = x^3 - 7x^2 + 17x - 15$ can be factored as

$$f(x) = (x^2 - 4x + 5)(x - 3).$$

Note that $x^2 - 4x + 5$ is irreducible in $\mathbb{R}[x]$, so this is the complete factorization of $f(x)$ in $\mathbb{R}[x]$. Meanwhile, the complete factorization of $f(x)$ in $\mathbb{C}[x]$ is

$$f(x) = \big(x - (2+i)\big)\big(x - (2-i)\big)\big(x - 3\big).$$

The location of the roots in the complex plane are visualized in the figure below.



7. Let $f(x) = x^4 - 3x^3 + 5x^2 - 3x + 4$. Verify that $f(i) = 0$ and use this fact to completely factorize $f(x)$ in $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

Solution. Note that $i^4 = -1$, $i^3 = -i$, and $i^2 - 1$, hence

$$f(i) = 1 + 3i - 5 - 3i + 4 = 0 + 0i = 0.$$

Hence i is a root of $f(x)$ and it follows that $-i$ is also a root of $f(x)$. Thus

$$(x - i)(x + i) = x^2 + 1$$

is a divisor of $f(x)$. We can use polynomial long division to find that

$$
\begin{array}{r|l}
 & x^2 - 3x + 4 \\
\hline
x^2 + 1 & x^4 - 3x^3 + 5x^2 - 3x + 4 \\
 & -x^4 \qquad\quad - x^2 \\
\hline
 & \quad -3x^3 + 4x^2 - 3x \\
 & \quad\;\; 3x^3 \qquad\; + 3x \\
\hline
 & \qquad\quad 4x^2 \qquad + 4 \\
 & \qquad -4x^2 \qquad - 4 \\
\hline
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

and thus we can factor $f(x)$ as

$$f(x) = (x^2 + 1)(x^2 - 3x + 4).$$

It remains to factor $x^2 - 3x + 4$. Note that $3^2 - 4 \cdot 4 = 9 - 16 = -7 < 0$, and thus this polynomial has no real roots. However, we can still use the quadratic equation to find the complex roots. The roots are of the form

$$x = \frac{3 \pm \sqrt{3^2 - 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm i\sqrt{7}}{2}.$$

So all of the roots of $f(x)$ are

$$i, \quad -i, \quad \frac{3 + i\sqrt{7}}{2}, \quad \text{and} \quad \frac{3 - i\sqrt{7}}{2}.$$

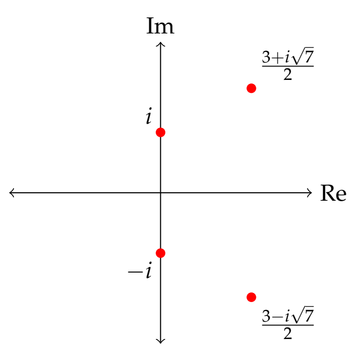
Hence the factorization of $f(x)$ is $\mathbb{C}[x]$ is

$$f(x) = (x - i)(x + i)\left(x - \frac{3 + i\sqrt{7}}{2}\right)\left(x - \frac{3 - i\sqrt{7}}{2}\right)$$

while the factorization of $f(x)$ in $\mathbb{R}[x]$ is

$$f(x) = (x^2 + 1)(x^2 - 3x + 4).$$

The location of the roots in the complex plane are visualized in the figure below.



8. Let $z, w \in \mathbb{C}$. Prove that $|z + iw| = |z - iw|$ if and only if $z\overline{w} \in \mathbb{R}$.

Solution. First note that $\overline{z\overline{w}} = \overline{z}w$, and so by Properties of the Conjugate (PCJ) we have

$$z\overline{w} - \overline{z}w = 2i \operatorname{Im}(z\overline{w}).$$

and thus

$$i(z\overline{w} - \overline{z}w) = -2 \operatorname{Im}(z\overline{w}).$$

Note that $\overline{iw} = -i\overline{w}$ and thus $\overline{z + iw} = \overline{z} - i\overline{w}$ by PCJ, and by the Properties of the Modulus (PM) we have

$$\begin{aligned} |z + iw|^2 = (z + iw)\overline{(z + iw)} &= (z + iw)(\overline{z} - i\overline{w}) \\ &= z\overline{z} + i\overline{z}w - iz\overline{w} + w\overline{w} \\ &= |z|^2 + |w|^2 - i(z\overline{w} - \overline{z}w) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Im}(z\overline{w}). \end{aligned}$$

Similarly, note that $\overline{z - iw} = \overline{z} + i\overline{w}$ and thus

$$\begin{aligned} |z - iw|^2 = (z - iw)\overline{(z - iw)} &= (z - iw)(\overline{z} + i\overline{w}) \\ &= z\overline{z} - i\overline{z}w + iz\overline{w} + w\overline{w} \\ &= |z|^2 + |w|^2 + i(z\overline{w} - \overline{z}w) \\ &= |z|^2 + |w|^2 - 2 \operatorname{Im}(z\overline{w}). \end{aligned}$$

Hence $|z + iw| = |z - iw|$ is satisfied if and only if $\operatorname{Im}(z\overline{w}) = -\operatorname{Im}(z\overline{w})$ which occurs if and only if $\operatorname{Im}(z\overline{w}) = 0$, which is equivalent to $z\overline{w} \in \mathbb{R}$.