MATH 135 — Fall 2021
Sample Proofs from Lecture 5

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Proving equalities

To prove a universally quantified equality ("$\forall x \in S,\ f(x) = g(x)$"):

- Start by letting the variables be arbitrary elements of the domain.
- Show that the LHS (left-hand side) is equal to the RHS (right-hand side) by writing out the expression of the LHS and manipulating it until you get the RHS.
- Make sure to explain or justify each each step that is not just a straightforward manipulation!

Claim. *For every $\theta \in \mathbb{R}$, it holds that*

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

Proof. First recall the following trigonometric identities. For every choice of real numbers $\alpha, \beta \in \mathbb{R}$, one has the following angle addition formula:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{$*$}$$

For every real number $\alpha \in \mathbb{R}$, one also has that

$$\sin^2\alpha = \frac{1 - \cos(2\alpha)}{2} \tag{$**$}$$

and that

$$\sin^2\alpha + \cos^2\alpha = 1. \tag{$***$}$$

Now let θ be an arbitrary real number. One has

$$\begin{aligned}
\sin(3\theta) &= \sin(2\theta + \theta) \\
&= \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta && \text{(by the angle addition formula in } (*)) \\
&= 2\sin\theta\cos^2\theta + \cos(2\theta)\sin\theta && \text{(again by } (*)) \\
&= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta && \text{(by } (**)) \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta && \text{(by rearranging)} \\
&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta && \text{(by } (***)) \\
&= 3\sin\theta - 4\sin^3\theta,
\end{aligned}$$

as desired. $\square$

Proving inequalities

Proving a universally quantified inequality ("$\forall x \in S,\ f(x) \geq g(x)$" or "$\forall x \in S,\ f(x) > g(x)$"):

- Same idea as for equalities, but at each step the expression needs to be either equal to or greater than the next expression.

Claim. *For every $x \in \mathbb{R}$ it holds that*

$$x^2 + 5x + 7 > 0.$$

Proof. Let x be a real number. Now

$$\begin{aligned} x^2 + 5x + 7 &= \left(x^2 + 5 + \frac{25}{4}\right) - \frac{25}{4} + 7 \\ &= \left(x + \frac{5}{2}\right)^2 + \frac{3}{4} \\ &\geq \frac{3}{4} && \text{(because all squares of real numbers are non-negative)} \\ &> 0, \end{aligned}$$

which completes the proof. $\square$

Proof by cases

You can sometimes prove a statement by:

1. Dividing the situation into cases which exhaust all the possibilities; and
2. Showing that the statement follows in all cases.

Claim. *For every choice of real numbers $x, y \in \mathbb{R}$, it holds that*

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}.$$

Proof. Let x and y be real numbers. There are two cases to consider: either $x \geq y$ or $x < y$.

Case 1: Suppose $x \geq y$ such that $x - y \geq 0$ and thus $|x - y| = x - y$. One has that

$$\max\{x, y\} = x = \frac{2x}{2} = \frac{x + y + x - y}{2} = \frac{x + y + |x - y|}{2},$$

as desired.

Case 1: Suppose $x < y$ such that $x - y < 0$ and thus $|x - y| = -x + y$. One has that

$$\max\{x, y\} = y = \frac{2y}{2} = \frac{x + y - x + y}{2} = \frac{x + y + |x - y|}{2},$$

as desired.

This proves the claim, as the equality has been shown to hold in every possible case. $\square$