MATH 135 — Fall 2021
Sample Proofs from Lecture 8

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Proof by contrapositive

A universally quantified statement of the form

$$\forall x \in \mathcal{S},\, P(x) \implies Q(x)$$

is logically equivalent to its contrapositive:

$$\forall x \in \mathcal{S},\, \neg Q(x) \implies \neg P(x).$$

For some universally quantified statements, it is easier to prove the contrapositive than to prove the original statement directly.

Claim. $\forall x \in \mathbb{R},\, x^3 - 5x^2 + 3x \neq 15 \implies x \neq 5.$

Proof. We prove the contraspositive. Let x be a real numbers and suppose that $x = 5$. Then

$$x^3 - 5x^2 + 3x = 5^3 - 5 \cdot 5^2 + 3 \cdot 5 = 5^3 - 5^3 - 15 = 15,$$

which completes the proof. □

Claim. *For all integers k, if $k^2 + 4k - 2$ is odd then k is odd.*

Proof. We prove the contrapositive. Let k be an integer and suppose that k is even. There exists an integer m such that $k = 2m$. Now,

$$k^2 + 4k - 2 = (2m)^2 + 4k - 2 = 2(2m^2 + 2k - 1),$$

which is even as $2m^2 + 2k - 1$ is an integer. □

Claim. *For all real numbers x and y, if xy is irrational then x is irrational or y is irrational.*

Symbollically, this claim can be written as:

$$\forall x, y \in \mathbb{R},\ xy \notin \mathbb{Q} \implies (x \notin \mathbb{Q} \vee y \notin \mathbb{Q}).$$

The contrapositive of this statement is

$$\forall x, y \in \mathbb{R},\ (x \in \mathbb{Q} \wedge y \in \mathbb{Q} \implies xy \in \mathbb{Q}).$$

Proof. We prove the contrapositive. Let x and y be real numbers ans suppose that both x and y are rational. There exist integers a, b, m, and n such that $b \neq 0$ and $n \neq 0$ and

$$x = \frac{a}{b} \quad \text{and} \quad y = \frac{m}{n}.$$

Now

$$xy = \frac{a}{b} \cdot \frac{m}{n} = \frac{am}{bn},$$

where am and bn are integers and $bn \neq 0$ as both b and n are nonzero. We conclude that xy is rational, which completes the proof. □

Proof by Method of Elimination

For sentences A, B, and C, it can be shown that

$$\big(A \implies (B \vee C)\big) \equiv \big((A \wedge \neg B) \implies C\big).$$

That is, to prove that either B or C is true, we can suppose B is false, which 'eliminates' the possibility of B being true, and then prove in this case that C must be true.

For universally quantified statements, this looks like:

$$\Big(\forall x \in S, \big(P(x) \implies (Q(x) \vee R(x))\big)\Big) \equiv \Big(\forall x \in S, \big((P(x) \wedge \neg Q(x)) \implies R(x)\big)\Big).$$

Claim. *For all real numbers x, if $|2x - 6| = 4$ then $x \geq 3$ or $x = 1$.*

Proof. We prove this statement by the Method of Elimination. Let x be a real number and suppose that $|2x - 6| = 4$. Suppose further that $x < 3$. Then $2x \leq 6$ and thus $2x - 6 \leq 0$ which implies that $|2x - 6| = 6 - 2x$. It follows that from the assumption that $|2x - 6| = 4$ that

$$6 - 2x = 4$$

and solving this equation for x yields $x = 1$, as desired. □

Proving "if and only if" statements

To prove a statement of the form $A \iff B$, one must prove both $A \implies B$ and $B \implies A$. For universally quantified statements, this equivalence is

$$\Big(\forall x \in S,\, P(x) \iff Q(x)\Big) \equiv \Big(\forall x \in S,\, (P(x) \implies Q(x)) \wedge (Q(x) \implies P(x))\Big).$$

> **Claim.** *Suppose x and y are real numbers such that $x \geq 0$ and $y \geq 0$. Then $\frac{x+y}{2} = \sqrt{xy}$ if and only if $x = y$.*

Proof. First suppose that $x = y$. Then

$$\frac{x+y}{2} = \frac{x+x}{2} = x = \sqrt{x \cdot x} = \sqrt{xy}.$$

Conversely, suppose instead that $\frac{x+y}{2} = \sqrt{xy}$. Multiplying both sides by 2 yields

$$x + y = 2\sqrt{xy}.$$

Squaring both sides, we find that

$$x^2 + 2xy + y^2 = 4xy,$$

which is equivalent to

$$x^2 - 2xy + y^2 = 0$$

and thus

$$(x-y)^2 = 0.$$

We conclude that $x - y = 0$ and thus $x = y$. This completes the proof. □

Note that $B \implies A$ is equivalent to $\neg A \implies \neg B$. Hence sometimes it is easier to prove

$$A \implies B \quad \text{and} \quad \neg A \implies \neg B,$$

which also proves equivalence.

> **Claim.** *For all integers a, one has that a is even if and only if a^2 is even.*

Proof. Let a be an integer. First suppose that a is even, such that there is an integer k satisfying $2k = a$. Thus

$$a^2 = 4k^2 = 2 \cdot (2k^2),$$

which is even as $2k^2$ is an integer. Conversely, suppose that a is odd such that there is an integer k satisfying $2k + 1 = a$. Thus

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1,$$

which is odd as $2k^2 + 2k$ is an integer. This completes the proof. □