MATH 135 — Fall 2021
Sample Proofs from Lecture 10

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Principle of Mathematical Induction

Let $P(1), P(2), P(3), \cdots$ be a sequence of statements. (I.e., suppose $P(n)$ is an open sentence for numbers $n \in \mathbb{N}$.) If the following are true:

(i) $P(1)$

(ii) $\forall k \in \mathbb{N},\ P(k) \implies P(k+1)$

then it is also true that

(iii) $\forall n \in \mathbb{N},\ P(n)$.

Example

Claim. $\forall n \in \mathbb{N},\ 4 \mid (5^n - 1)$.

Proof. We prove by induction. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $4 \mid (5^n - 1)$.

- Base case: When $n = 1$, we have

  $$5^n - 1 = 5 - 1 = 4,$$

  which is divisible by 4, so $P(1)$ is true.

- Induction step: Let k be an arbitrary natural number and suppose that $P(k)$ is true. That is, assume that there exists an integer m such that

  $$4m = 5^k - 1. \qquad \text{(IH)}$$

  Now,

  $$\begin{aligned} 5^{k+1} - 1 &= 5(5^k) - 1 \\ &= 5(4m+1) - 1 && \text{(by IH)} \\ &= 4(5m+1), \end{aligned}$$

  which is divisible by 4 as $5m + 1$ is an integer, and thus $P(k+1)$ is true.

By the principle of mathematical induction, it holds that $4 \mid (5^n - 1)$ for every $n \in \mathbb{N}$. □

Example

> **Claim.** *For every positive integer n, it holds that*
> $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. We prove by induction. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

- Base case: When $n = 1$, we have

  $$\sum_{i=1}^{n} i(i+1) = 1(1+1) = 2 = \frac{1 \cdot 2 \cdot 3}{3},$$

  and thus $P(1)$ is true.

- Induction step: Let k be an arbitrary natural number and suppose that $P(k)$ is true. That is, assume that

  $$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}. \tag{IH}$$

  Now,

  $$\begin{aligned}\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) && \text{(by IH)} \\ &= (k+1)(k+2)\left(\frac{k}{3}+1\right) \\ &= \frac{(k+1)(k+2)(k+3)}{3}\end{aligned}$$

  and thus $P(k+1)$ is true.

By the principle of mathematical induction, it holds that $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for every $n \in \mathbb{N}$. □

Base case does not have to be $n = 1$

> **Claim.** *For all $n \in \mathbb{N}$, if $n \geq 4$ then $n! > 2^n$.*

Proof. We prove by induction. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $n! > 2^n$.

- Base case: When $n = 4$, we have

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 > 16 = 2^4,$$

  and thus $P(4)$ is true.

- Induction step: Let k be an arbitrary natural number such that $k \geq 4$. Suppose that $P(k)$ is true. That is, assume that

$$k! > 2^k. \tag{IH}$$

  Now,

$$\begin{aligned}
(k+1)! &= k!(k+1) \\
&> 2^k(k+1) && \text{(by IH)} \\
&\geq 2^k \cdot 5 && \text{(because } k \geq 4\text{)} \\
&> 2^k \cdot 2 && \text{(because } 5 > 2\text{)} \\
&= 2^{k+1}
\end{aligned}$$

  and thus $(k+1)! > 2^{k+1}$, so $P(k+1)$ is true.

By the principle of mathematical induction, it holds that $n! > 2^n$ for every $n \geq 4$. □