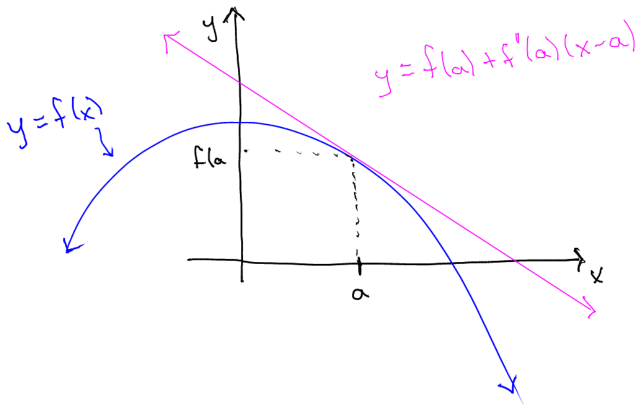
§ 3.5 Linear Approximations

Let f be a differentiable function at a point a. Consider the tangent line to the graph of f at a. We already found the equation that defines this line: $y = f(a) + f'(a)(x-a)$.



The tangent line itself can be considered as the graph of the function

$$g(x) = f(a) + f'(a)(x-a).$$

This function has the following properties:

- g is linear
- $g(a) = f(a)$
- $g'(a) = f'(a)$

Moreover, it is the unique linear function with $g(a) = f(a)$ and $g'(a) = f'(a)$.

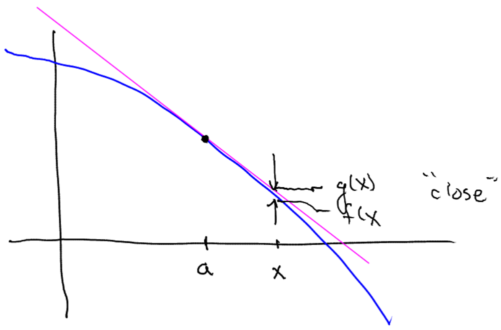
Note that:

$$g'(x) = \frac{d}{dx}\left(f(a) + f'(a)(x-a)\right)$$

$$= \underbrace{\frac{d}{dx} f(a)}_{0} + \underbrace{f'(a)}_{\text{constant}} \underbrace{\frac{d}{dx}(x-a)}_{1}$$

$$= f'(a)$$

Now $g(a) = f(a)$, but what about $g(x)$ for values of x that are "close" to a? From the picture above, we see that $g(x)$ is "close" to $f(x)$ if x is close to a:



That is, $g(x)$ is a good approximation for the value of $f(x)$:

$$g(x) \approx f(x)$$

"approximately equal to"

Definition If f is differentiable at a, the linear approximation of f at a is the function L_a^f defined by

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

for all $x \in \mathbb{R}$.

Note: If x is close to a we can write $x = a + \delta$ for some small δ (delta). Then $x - a = \delta$ and we can write

$$L_a^f(a+\delta) = f(a) + f'(a)\,\delta.$$

That is, the value of the linear approximation depends only on the displacement δ away from the point a of linearization.

Application

Linear approximations are useful for approximating values:

Example

Use linear approximations to estimate the value of $\sqrt{3.98}$.

Solution: Note that 3.98 is "close" to 4 so we should expect $\sqrt{3.98}$ to be "close" to $\sqrt{4} = 2$.

Let $f(x) = \sqrt{x}$ and $a = 4$. Note that $3.98 = 4 - 0.02 = a + \delta$ where $\delta = -0.02$. At $a = 4$, $f(4) = 2$ and $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(4) = \frac{1}{4}$.

So the linear approximation at $a = 4$ is

$$\begin{aligned} L_4^f(x) &= f(4) + f'(4)(x-4) \\ &= 2 + \tfrac{1}{4}(x-4) \\ &= 1 + \tfrac{x}{4}. \end{aligned}$$

Now we approximate:

$$\begin{aligned} \sqrt{3.98} = f(3.98) &\approx L_4^f(4-0.02) \\ &= f(4) + f'(4)(-0.02) \\ &= 2 - \frac{0.02}{4} \\ &= 2 - \frac{0.01}{2} = 2 - 0.005 \\ &= 1.995 \end{aligned}$$

The "true value" is $\sqrt{3.98} = 1.9949937\ldots$, so we're not far off!

The error in our estimate is

$$|f(3.98) - L_4^f(3.98)| = |1.9949937\ldots - 1.995| = 0.0000063$$

Approximate the value of $(0.99)^{10}$.

Solution: Here we note that 0.99 is close to 1, so $(0.99)^{10}$ should be close to $1^{10} = 1$. We use the function $f(x) = x^{10}$ and set $a = 1$.

We have $f(1) = 1$ and $f'(x) = 10x^9$ so that $f'(1) = 10$. The linear approximation of f at $a = 1$ is

$$L_1^f(x) = 1 + 10(x-1).$$

At $x = 0.99 = 1 - \frac{1}{100}$, we have

$$\begin{aligned} (0.99)^{10} = \left(1 - \tfrac{1}{100}\right)^{10} = f\left(1 - \tfrac{1}{100}\right) &\approx L_1^f\left(1 - \tfrac{1}{100}\right) \\ &= 1 + 10\left(-\tfrac{1}{100}\right) = 1 - \tfrac{1}{10} = 0.9 \end{aligned}$$

So $(0.99)^{10} \approx 0.9$. "True" answer is $(0.99)^{10} = 0.904382075\ldots$

and thus the error in our estimate is

$$|f(0.99) - L_1^f(0.99)| = |0.90438\ldots - 0.9|$$
$$= 0.00438\ldots$$

Note that these approximations are only good if $|x-a|$ is "small".

For example, if we use the linear approximation for the square root function at $a=4$ to estimate the value of $\sqrt{9}$, we'd get

$$3 = \sqrt{9} = \sqrt{4+5} = f(4+5) \approx L_4^f(4+5)$$
$$= f(4) + f'(4)\,5$$
$$= 2 + \tfrac{1}{4}\,5$$
$$= 3.25$$

Not a great approximation for 3.

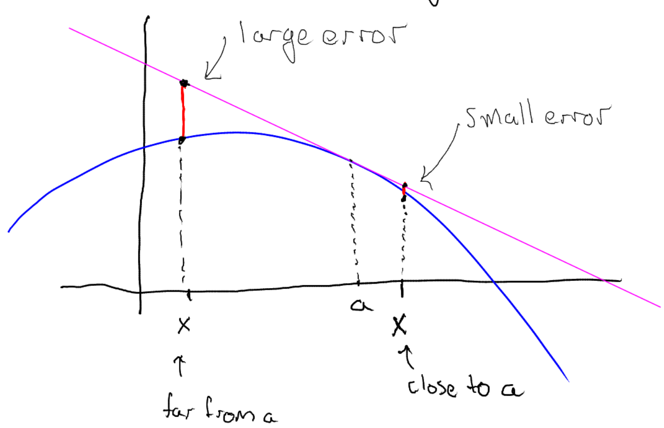
How can we formalize how "good" an approximation is?

Without knowing an upper bound on the error, the approximation is useless!

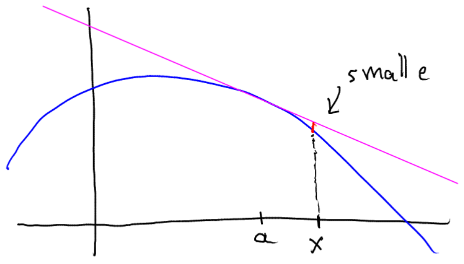
We need a way to get an upper bound on the error.

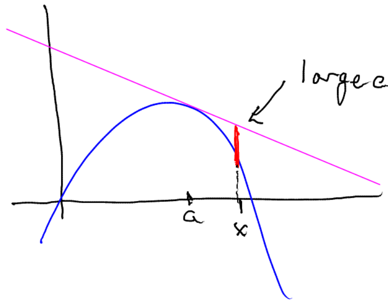
There are a few things that affect how "good" an approximation is.

- The farther x gets away from a, the less sure we can be about how good the approximation is.



- How "curved" the graph of f is at a.





What property of the function tells us how "curved" the graph is at a? The second derivative! $f''(a)$.

Ex: Linear function:

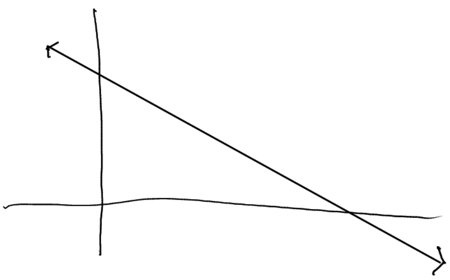
If $f(x) = mx + b$ then the linear approximation is just

$L_a^f(x) = mx + b$. So error in approximation is always zero:

$$|f(x) - L_a^f(x)| = 0$$

$f'(x) = m$

$f''(x) = 0$ ← second derivative is zero, so no curvature!



Ex: Consider $f(x) = 2 - x^2$ and $g(x) = 2 - 2x^2$.

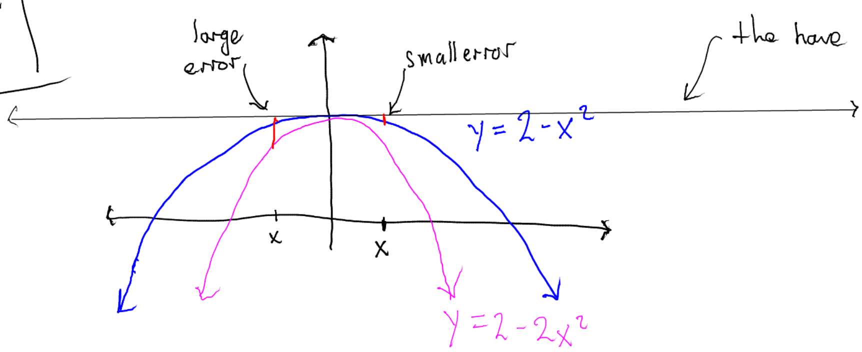
Then $f'(x) = -2x$ and $g'(x) = -4x$

$f''(x) = -2$ $\qquad g''(x) = -4$.

Magnitude of second derivatives: $|f''(x)| = 2$

$|g''(x)| = 4$ $\qquad$ So g is more "curved"

More curvature ⇒ more error



the have the same Linear approximation at $a = 0$, but the linear approximation is much worse for g than for f.

How can we use this to get bounds on the error?

Theorem (Error in linear approximation)

Let $a \in \mathbb{R}$ and let I be an interval containing a. Suppose that f is twice differentiable(*) on I and let $M \geq 0$ be a number such that

$$|f''(x)| \leq M \quad \text{for all } x \in I.$$

Then
$$\boxed{\underbrace{|f(x) - L_a^f(x)|}_{\text{Actual error}} \leq \underbrace{\frac{M}{2}(x-a)^2}_{\text{upper bound on error}}} \quad \text{for all } x \in I.$$

(*) **Note**: f is twice differentiable if f' and f'' exist on all of I.

Idea: If we can find upper bound on the curvature $|f''(x)|$ then we can get useful upper bounds on the error of the linear approximation.

We won't prove this (although we'll use similar ideas later in the course), but we will do an example.

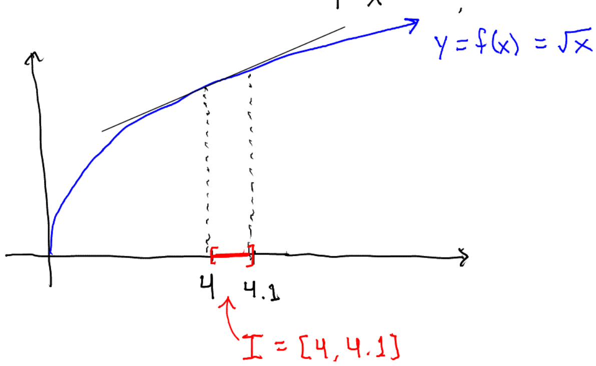
Ex Approximate the value of $\sqrt{4.1}$ and use the theorem to get upper bound on the error.

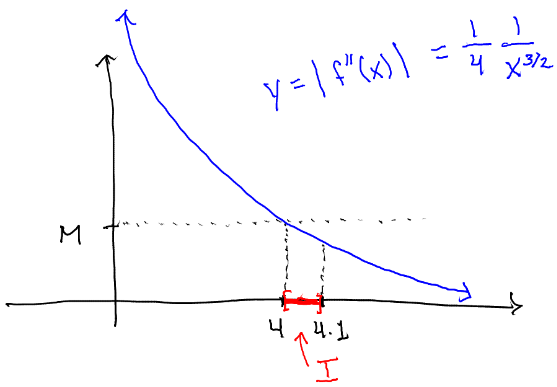
Solution:

Consider the interval $I = [4, 4.1]$ and let $a = 4$ and $f(x) = \sqrt{x}$.

The second derivative is computed as:
$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f''(x) = -\frac{1}{4}x^{-3/2}$$

So $|f''(x)| = \frac{1}{4}\frac{1}{x^{3/2}}$





Note that $|f''(x)|$ is decreasing, so $|f''(4)| \geq |f''(x)|$ for all $x \in I$.

Set $M = |f'(4)|$
$$= \frac{1}{4}\frac{1}{4^{3/2}} = \frac{1}{4^{5/2}} = \left(\frac{1}{4^{1/2}}\right)^5$$
$$= \frac{1}{2^5} = \frac{1}{32}$$

So $M = \frac{1}{32}$.

With this $M > 0$, we have $|f''(x)| \leq M$ for all $x \in I$.

The linear approximation of $f(x) = \sqrt{x}$ at $a = 4$ is

$$L_4(x) = 2 + \frac{1}{4}(x-4) = 1 + \frac{x}{4}.$$

And thus $\sqrt{4.1} \approx L_4(4.1) = 2 + \frac{0.1}{4}$

$$= 2 + 0.025 = 2.025$$

From the Theorem, we know the error is bounded by

$$\left|\sqrt{4.1} - 2.025\right| \leq \frac{M}{2}(4.1-4)^2 = \frac{1}{64}(0.1)^2$$

$$= \frac{1}{6400}$$

$$= 0.00015625.$$

Thus, the true value of $\sqrt{4.1}$ is in the range

$$2.025 - \frac{1}{6400} \leq \sqrt{4.1} \leq 2.025 + \frac{1}{6400}$$

$$\Rightarrow \boxed{2.02484 \leq \sqrt{4.1} \leq 2.025156}$$

Applications of Linear Approximations

- Suppose variables x and y are related by $y = f(x)$ for some function f.
- If x starts at a and changes a little bit, approximately how much does y change?

Using $f(x) \approx L_a^f(x) = f(a) + f'(x)(x-a)$ for x near a,

we take $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$ so that

$$\boxed{\Delta y \approx f'(a)\,\Delta x.} \qquad \boxed{\Delta y \approx \frac{dy}{dx}\,\Delta x}$$

Example.

Imagine you are inflating a spherical baloon with a pump.
At one point, the radius of the spherical baloon is exactly 10m.
After one more pump, the radius increases by 1 cm. (I.e, new

radius is $r = 10.01\,m$).

Q: How much does the volume of the baloon change (approximately)?

Ans: Volume and radius related by $V = \frac{4}{3}\pi r^3$.

$$\frac{dV}{dr} = 4\pi r^2$$

$$\Delta V \approx \underbrace{\left(\frac{dV}{dr}\right)\Big|_{r=10m}} \Delta r$$

$$= 4\pi(10m)^2 \; 0.01\,m$$

$$= 4\pi\left(100 \cdot \frac{1}{100}\right) m^3 \qquad = 4\pi \; m^3$$

$$\approx 12.6\,m^3$$

$\Rightarrow$ Volume changes by $\sim 12.6\,m^3$ when radius changes by 1cm

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§3.6 Newton's Method

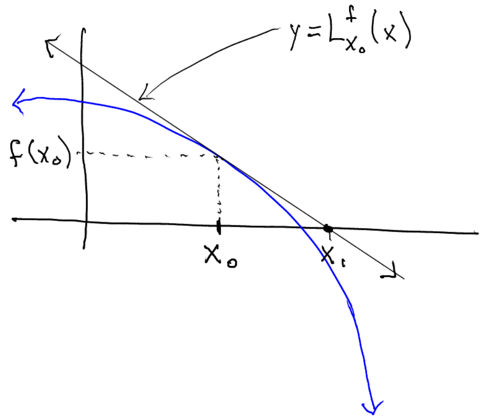
One method to find roots of a function: Bisection Method.

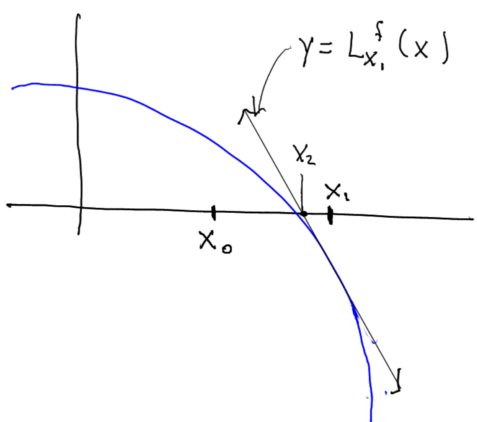
[Uses the fact that a function is continuous to determine root location using Intermediate Value Theorem.]

Another way is Newton's Method which uses the derivative and converges much faster. (The function must be differentiable.)

Idea: To find x so that $f(x) = 0$: Newton's Iterative Procedure

- Start with inital guess x_0.
- To get next value, find intersection of tangent line of f at a with the x-axis.





- That is, x_1 is the number such that

$$L^f_{x_0}(x_1) = 0. \qquad \text{where } L^f_{x_0} \text{ is linear approximation at } x_0.$$

$$L^f_{x_0}(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) = 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Then find the next point x_2 to be the number such that

$$L^f_{x_1}(x_2) = 0.$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- Continue... given x_n find $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

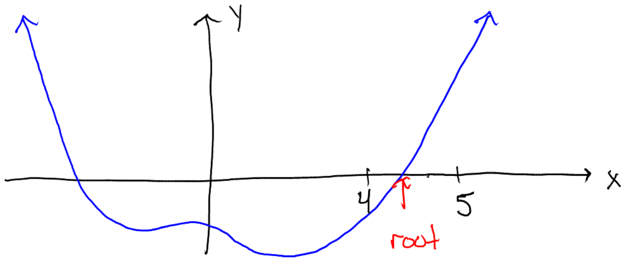
- The sequence $x_0, x_1, x_2, \ldots$ should converge to a <u>root</u> of f.

<u>Example</u> Use Newton's Method to find the root of the function defined by

$$f(x) = 3x^4 + 15x^3 - 125x - 1500$$

between 4 and 5, with error at most 10^{-5}.

<u>Sol</u>: First let's plot a graph (using graphing software)



<u>Note</u>: $f(4) = -272$ and $f(5) = 1625$
$\quad\quad\quad\quad\quad < 0 \quad\quad\quad\quad\quad\quad > 0$

so there is a root between 4 and 5 (by IVT).

Now $f'(x) = 12x^3 + 45x^2 - 125$

Set $x_0 = 4$.

For $n \in \mathbb{N}$, set $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

using calculator

$$x_1 = 4 - \frac{3\cdot(4)^4 + 15\cdot(4)^3 - 125\cdot(4) + 1500}{12\cdot(4)^3 + 45\cdot(4)^2 - 125} \approx 4.19956$$

$x_2 \approx 4.187268$

$x_3 \approx 4.187218711$

$x_4 \approx \underbrace{4.18721}8710$

to 5 decimal places, root is approximately $\underline{4.18722}$.

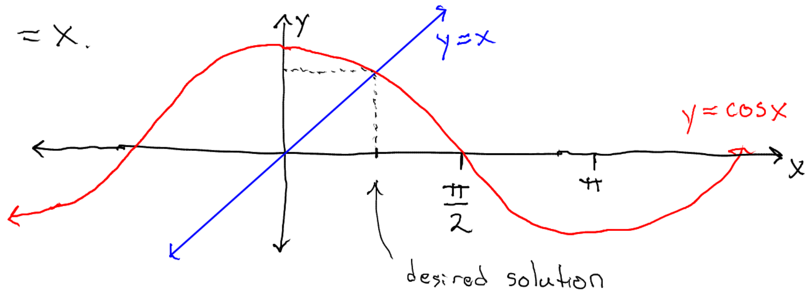
Can check that $f(4.18721) > 0 > f(4.18723)$

which verifies that root is between 4.18721 and 4.18723.

We can also use Newton's Method to obtain approximate solutions to equations.

<u>Ex</u>: Approximate the solution to the equation $\cos(x) = x$ to at least 8 decimal places.

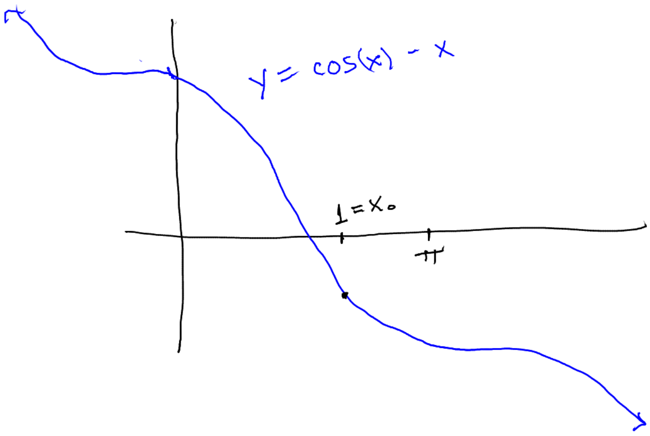
<u>Solution</u> Let's first examine a picture of the curves defined by $y = \cos x$ and $y = x$.



To find the point of intersection, we may define a function f as

$$f(x) = \cos(x) - x$$

and use Newton's Method to look for a root.



Let's first choose $x_0 = 1$ and evaluate

$$f(1) = \cos(1) - 1$$
$$= -0.4596976$$

(we'll need a calculator for this)

to get the next approximation,

$$f'(x) = -\sin(x) - 1.$$

Once we find the x_n term, the next term is computed by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n + \frac{\cos(x_n) - x_n}{\sin(x_n) + 1}$$

and $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = x_0 + \dfrac{\cos(x_0) - x_0}{\sin(x_0) + 1}$

$$= 1 + \frac{\cos(1) - 1}{\sin(1) + 1} = 0.7503638679$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \cdots$$

$$= 0.7391128909$$

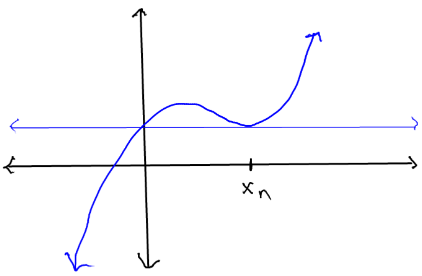
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \underline{0.739085133}4$$

$$x_4 = \cdots = \underline{0.739085133}2$$

Now we've got two approximations sharing the first 9 digits of the decimal expansion, so we can stop after only 4 steps.

When Newton's Method Goes Wrong

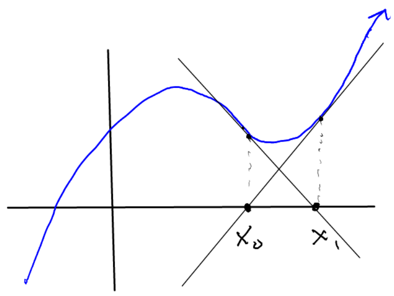
- If we reach a point where $f'(x_n) = 0$, then the tangent line to the graph at that point is parallel to the x-axis and never intersects, can't find next point!



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

can't divide by zero!

- It's also possible to "bounce" around a local minimum forever and never converge on the root.



$x_2 = x_0$

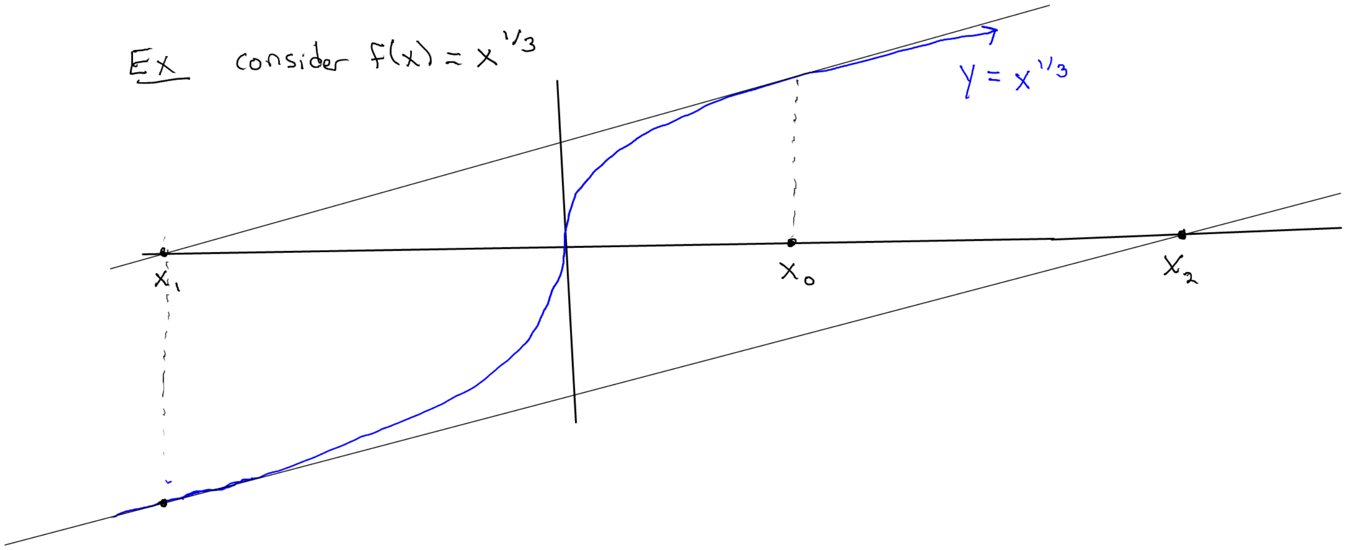
$x_3 = x_1$

$x_4 = x_0$

$\vdots$

- Also, if the graph of f is too "flat" it's possible for the sequence to diverge to infinity!

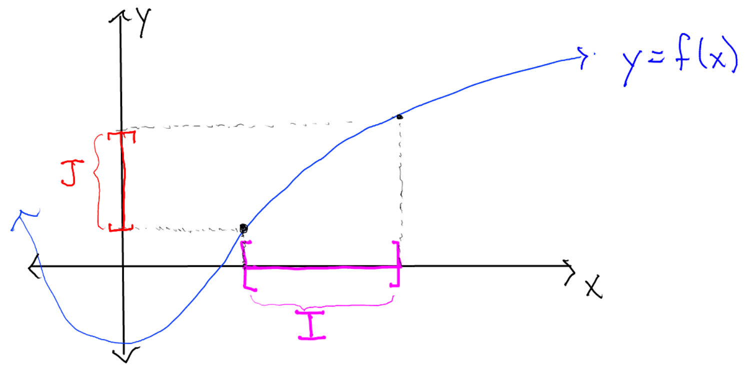
Ex consider $f(x) = x^{1/3}$

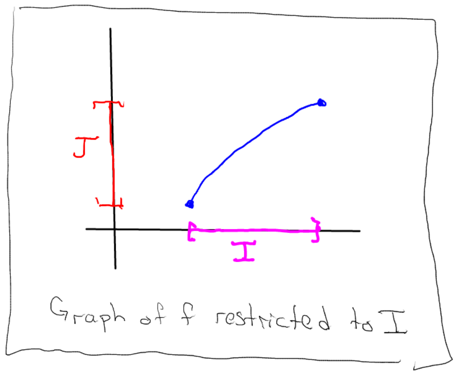


Fix: In all these cases, we picked our starting point x_0 "too far" from the actual root. Choosing a new x_0 closer to the root will fix the problem.

§3.10 Derivatives of Inverse Functions

Suppose we knew that f was <u>differentiable</u> and <u>invertible</u> on some interval I.



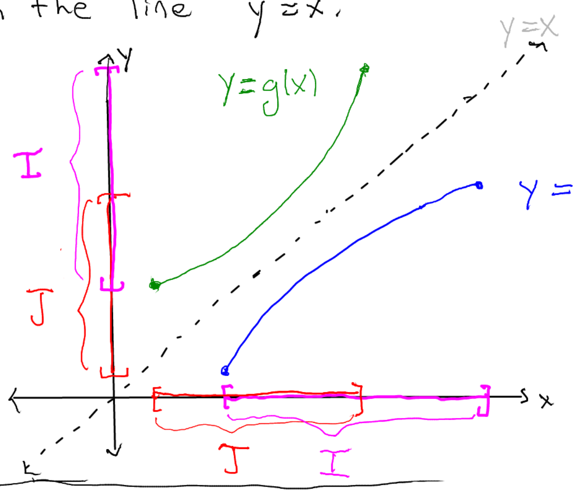


For example, in the picture above, the function is invertibe on I (even though it is not invertible on all of $\mathbb{R}$).

- Since f is continuous, the <u>range</u> of f is some interval J.
- There is some function g on J that is the inverse of f on I.

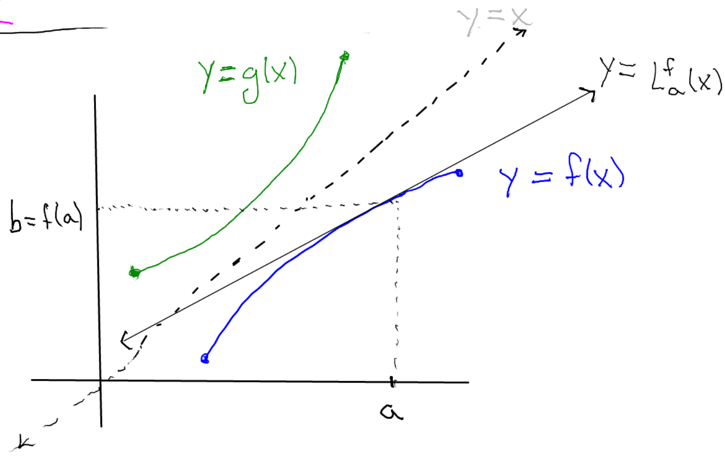
  That is:
  $$f(g(x)) = x \quad \text{for all } x \in J$$
  $$\text{and } g(f(x)) = x \quad \text{for all } x \in I.$$

  (I.e. g is the <u>inverse</u> of f)
- The graph of g is obtained from inverting the graph of f through the line $y = x$.

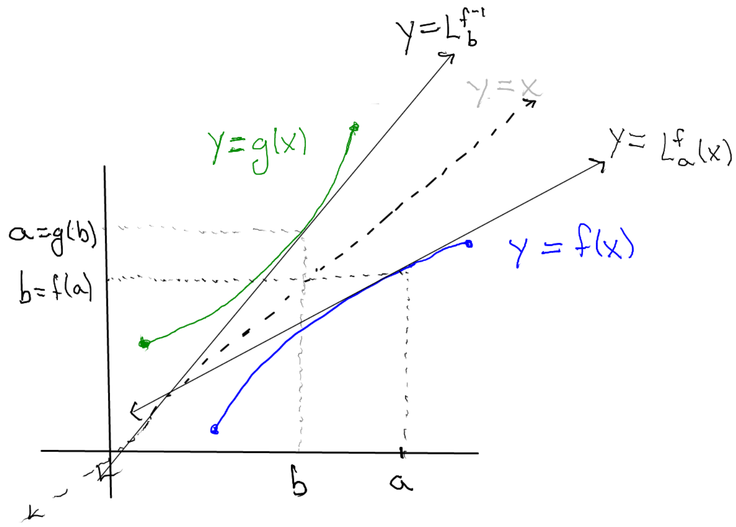


- Now, at a point $a \in I$, consider the tangent line to f at a, which has slope $f'(a)$:

  $$y = L^f_a(x)$$
  $$= f(a) + f'(a)(x - a).$$



• Setting $b = f(a)$ (so that $g(b) = g(f(a)) = a$), the mirror image of the tangent line to f at a gives us the tangent line to g at b.



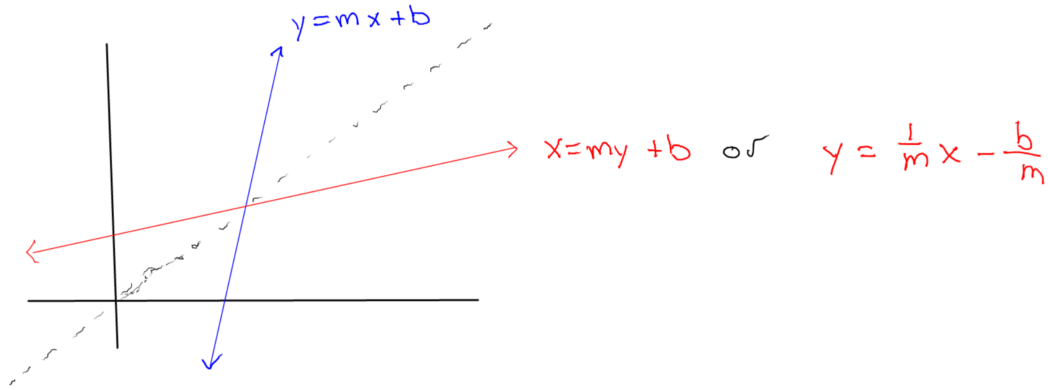
Q: What is the slope of this inverted line?

First consider what happens to an arbitrary line when it gets inverted through the $y=x$ line.

If we have a line $y = mx + b$ with slope m, we obtain the line inverted through the $y=x$ line by swapping the x and y.

The equation of the mirror image (red) line is therefore $x = my + b$.

Solving for y, we obtain $y = \frac{1}{m}x - \frac{b}{m}$.



Thus the slope of the reflected line is $\frac{1}{m}$.

A: The slope of the reflected line (i.e. the tangent line to g at b) is therefore $g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}$. ← Allowable as long as $f'(a) \neq 0$.

In particular, if f is differentiable then so is its inverse g.

We recap this discussion in the following theorem.

Theorem (Inverse Function Theorem)

Suppose f is invertible on some interval I with inverse function g. If f is differentiable at $a \in I$ and $f'(a) \neq 0$ then g is differentiable at $b = f(a)$ with derivative given by:

$$\boxed{g'(b) = \frac{1}{f'(g(b))}}.$$

For an arbitrary function f we typically denote its inverse by f^{-1} such that

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ where } f \text{ is defined}$$

and

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ where } f^{-1} \text{ is defined.}$$

We can also prove the inverse function theorem using the chain rule.

Proof: For all x in the range of f (i.e. all x where g is defined) we have

$$f(g(x)) = x$$

and thus $(f \circ g)(x) = x$. Taking the derivative of both sides,

$$\underbrace{\frac{d}{dx}\big((f\circ g)(x)\big)}_{} = \underbrace{\frac{d}{dx} x}_{=1}$$

$$= (f\circ g)'(x)$$

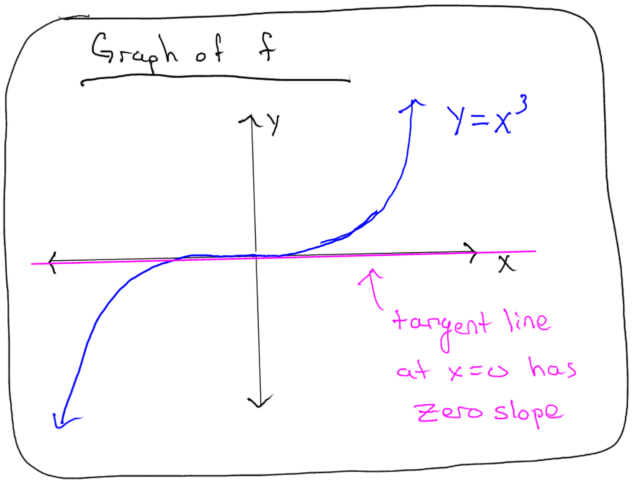
$$= f'(g(x))\, g'(x)$$

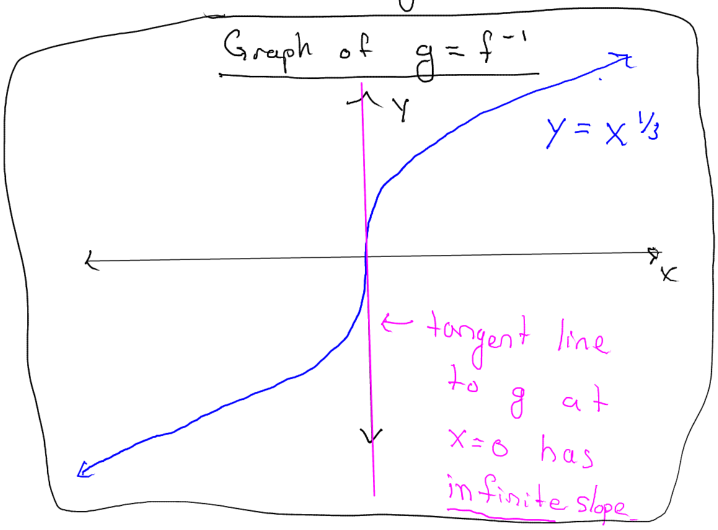
and thus $\boxed{g'(x) = \dfrac{1}{f'(g(x))}}$. □

Note that if $f'(a) = 0$ then g is not differentiable at $b = f(a)$.

This is because the tangent line to g at $b = f(a)$ will have "infinite slope".

For example, consider $f(x) = x^3$ which has inverse $g(x) = x^{1/3}$.





This makes sense from our differentiation rules as well. For $g(x) = x^{1/3}$, we have

$$g'(x) = \frac{1}{3} x^{-2/3} \quad \text{or} \quad g(x) = \frac{1}{3x^{2/3}}$$

which is defined everywhere except $x = 0$. Thus g is not differentiable at $x = 0$.

Example $\ln$ and $\log_a$

$\ln$: Note that if $f(x) = e^x$ then $\ln$ is inverse of f.

$\ln(e^x) = x$ for all $x \in \mathbb{R}$ and $e^{\ln x} = x$ for all $x > 0$

$$\ln'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

So $\boxed{\ln'(x) = \frac{1}{x}}$ $\quad \left[\frac{d}{dx}\ln(x) = \frac{1}{x}\right]$

$\log_a$: Also if $f(x) = a^x$ then $f'(x) = \ln(a)\, a^x$ and $\log_a$ is the inverse $\log_a(f(x)) = \log_a(a^x) = x$.

Thus
$$\log_a'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$
$$= \frac{1}{\ln(a)\, a^{\log_a(x)}}$$
$$= \frac{1}{\ln(a)\, x} = \frac{1}{\ln(a)} \frac{1}{x}$$

So $\boxed{\log_a'(x) = \frac{1}{\ln(a)\, x}}$ $\quad \frac{d}{dx}\log_a(x) = \frac{1}{\ln(a) x}$

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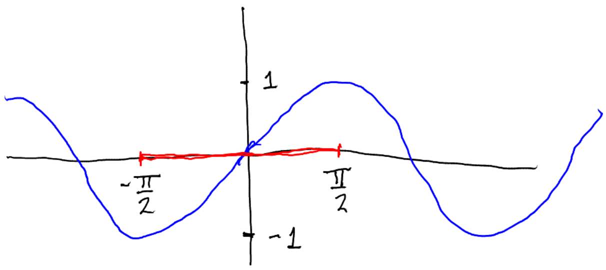
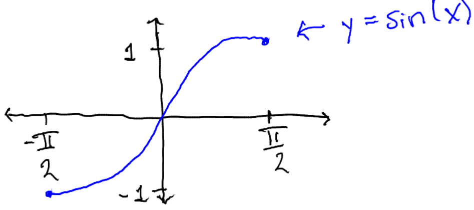
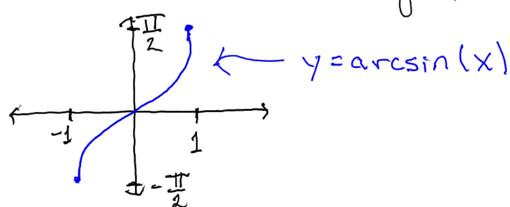
§3.11 Derivatives of Inverse Trig Functions

Let's first examine $\sin(x)$. Note that $\sin$ is not invertible on all of $\mathbb{R}$, but we can find an interval I on which $\sin$ is invertible.



If we take $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$, on this interval $\sin$ is invertible and looks like: And has an inverse whose graph is:





Note: For each $x \in [-1, 1]$ there is exactly one $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(y) = x$, and we write $\boxed{\arcsin(x) = y.}$

So arcsin has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

On $(-1, 1)$, what is $\arcsin'$?

$$\sin(\arcsin(x)) = x$$

$$\frac{d}{dx}\left[\sin(\arcsin(x))\right] = \underbrace{\frac{d}{dx} x}_{1}$$

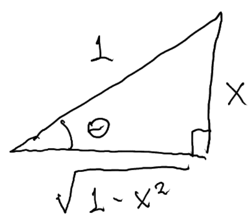
$$\sin'(\arcsin(x)) \arcsin'(x) = 1$$

$$\Rightarrow \cos(\arcsin(x)) \arcsin'(x) = 1$$

$$\Rightarrow \arcsin'(x) = \frac{1}{\cos(\arcsin(x))}$$

Can we simplify $\cos(\arcsin(x))$?

For a given x, let $\theta = \arcsin(x)$. We can draw the corresponding right triangle:



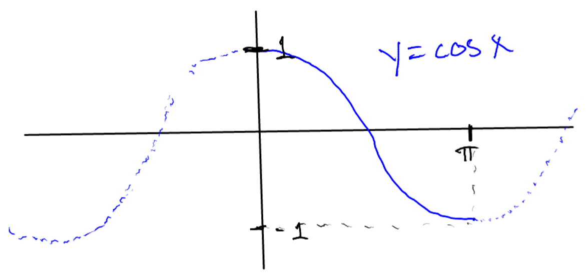
So that $\sin\theta = x$

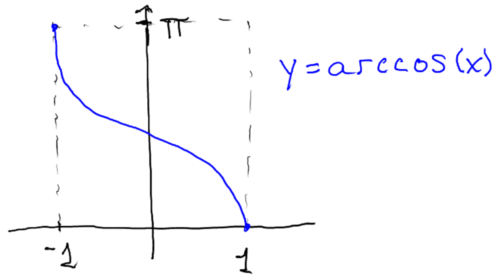
and $\cos\theta = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \cos(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$

Thus: $$\boxed{\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}}$$

- cos is invertible on $[0, \pi]$, and has inverse on this interval





On $(-1, 1)$ arccos is differentiable and

$$\cos(\arccos(x)) = x$$

$$\frac{d}{dx}\cos(\arccos(x)) = \underbrace{\frac{d}{dx}x}_{1}$$

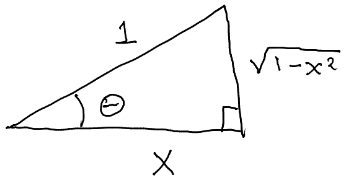
$$-\sin(\arccos(x))\arccos'(x) = 1$$

$$\Rightarrow \quad \arccos'(x) = -\frac{1}{\sin(\arccos(x))}$$

Now let $\theta = \arccos(x)$

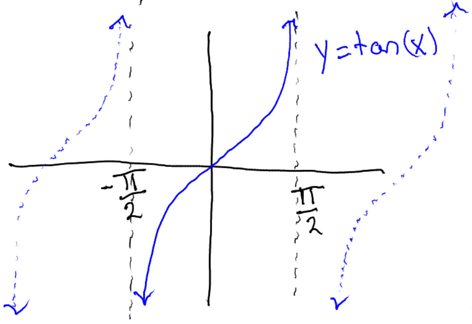
So that $\cos(\theta) = x$

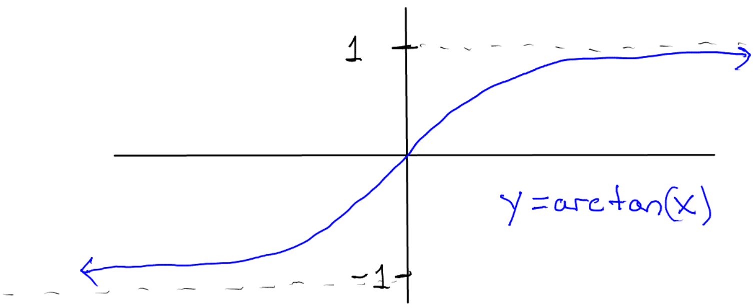
and $\sin(\theta) = \sqrt{1-x^2}$



$$\Rightarrow \quad \boxed{\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}}$$

On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ tan is invertible and has inverse





So arctan is differentiable on $\mathbb{R}$ with

$$\tan(\arctan(x)) = x$$

$$\Rightarrow \quad \frac{d}{dx}\tan(\arctan(x)) = \underbrace{\frac{d}{dx}x}_{1}$$

$$\Rightarrow \quad \tan'(\arctan(x))\arctan'(x) = 1$$

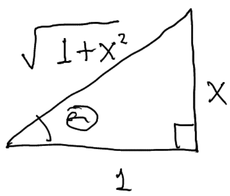
$$\Rightarrow \quad \arctan(x) = \frac{1}{\sec^2(\arctan(x))} \qquad \left(\text{since } \tan'(y) = \sec^2(y)\right)$$

$$= \cos^2(\arctan(x))$$

Now let $\theta = \arctan(x)$

so that $\tan(\theta) = x$

and $\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$



so $\boxed{\arctan'(x) = \frac{1}{1+x^2}}$

Examples: find f' if

- $f(x) = \arctan(e^{\sin x})$

$$f'(x) = \arctan'(e^{\sin x}) \frac{d}{dx}[e^{\sin x}]$$

$$= \frac{1}{1+(e^{\sin x})^2} e^{\sin x} \cos x$$

$$= \boxed{\frac{\cos x \, e^{\sin x}}{1+e^{2\sin x}}}$$

- $f(x) = \ln(\arctan x)$

$$f'(x) = \ln'(\arctan(x)) \arctan'(x)$$

$$= \boxed{\frac{1}{\arctan(x)} \frac{1}{1+x^2}}$$

- $f(x) = \arcsin(x) + \arccos(x)$

$$f'(x) = \arcsin'(x) + \arccos'(x)$$

$$= \frac{1}{\sqrt{1-x^2}} + -\frac{1}{\sqrt{1-x^2}} = 0$$

So $\arcsin(x) + \arccos(x)$ is constant (where it is defined)