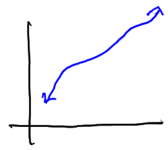
Week 9

§ 4.2.2 Increasing/Decreasing Functions

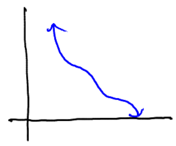
Definition

Let f be a function on an interval I.
We say that f is:

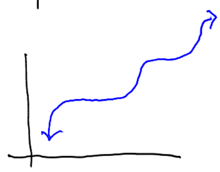
- (strictly) increasing if, for all $x_1, x_2 \in I$
  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$



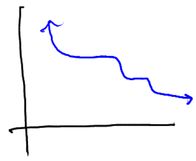
- (strictly) decreasing if, for all $x_1, x_2 \in I$
  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$



- non-decreasing if for all $x_1, x_2 \in I$
  $$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$



- non-increasing if for all $x_1, x_2 \in I$
  $$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$



The derivative of a function can give us information about where the function is increasing/decreasing.

Theorem (Increasing/Decreasing Function Theorem)

Suppose f is differentiable on an interval I and let $x_1, x_2 \in I$ such that $x_1 < x_2$.

1) If $f'(x) > 0$ for every $x \in I$ then $f(x_1) < f(x_2)$. (i.e. f is increasing)
2) If $f'(x) < 0$ for every $x \in I$ then $f(x_1) > f(x_2)$ (i.e. f is decreasing)
3) If $f'(x) \geq 0$ for every $x \in I$ then $f(x_1) \leq f(x_2)$. (i.e. f is non-decreasing)
4) If $f'(x) \leq 0$ for every $x \in I$ then $f(x_1) \geq f(x_2)$. (i.e. f is non-increasing)

The proof of this theorem makes use of the Mean Value Theorem, so let's restate that here for convenience.

Mean Value Theorem

Suppose f is

- differentiable on (a,b), and
- continuous on $[a,b]$.

There is a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof (of Inc/Decr. Function Thm):

We'll prove (1). The others are analogous.
Suppose that $f'(x) > 0$ holds for all $x \in I$. We now apply the MVT to the interval $[x_1, x_2]$. There is a point $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

By assumption, it holds that $f'(c) > 0$. Therefore $0 < \frac{f(x_2)-f(x_1)}{x_2-x_1}$ and thus $0 < f(x_2) - f(x_1)$ since $x_2 - x_1 > 0$. It follows that $f(x_1) < f(x_2)$.

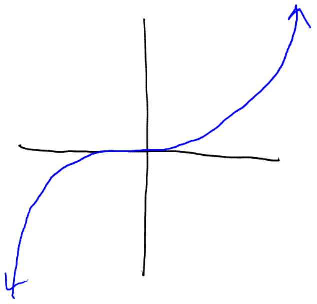
$\square$

Part (1) of the Incr./Decr. Function Theorem says that, for a function f,

$$\boxed{f' > 0 \Rightarrow f \text{ is increasing}}$$

Note that the converse is NOT true.

Example: Consider the function f defined as $f(x) = x^3$ for all $x \in \mathbb{R}$.



This function is strictly increasing everywhere (since $x < y \Rightarrow x^3 < y^3$) but $f'(0) = 0$.

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§4.2.3 Functions with Bounded Derivative

Q: Suppose you are driving in a car on a road where the speed limit is 100km/hr. If you never exceed the speed limit, what is the furthest possible distance you could travel in 1 hour?

Ans: 100km!

Idea: If your position is a function of time $f(t)$, the speed is the derivative $f'(t)$. Placing bounds on the derivative allows us to place bounds on the original function.

Theorem: (Bounded Derivative Theorem)
Suppose f is differentiable on (a,b) and continuous on $[a,b]$. Let $m, M \in \mathbb{R}$ and suppose $m \leq f'(x) \leq M$ for every $x \in (a,b)$. Then

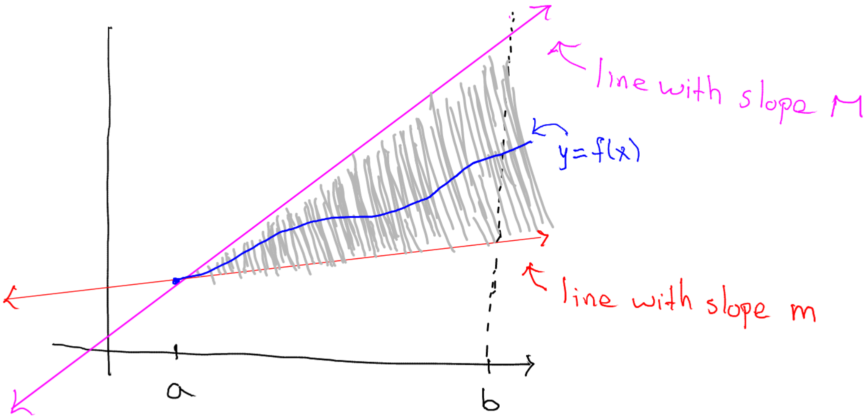
$$f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a)$$

for every $x \in [a,b]$.

Idea: The lines defined by

$$y = f(a) + m(x-a) \quad \text{and} \quad y = f(a) + M(x-a)$$

Provide upper and lower bounds for $f(x)$ for $x \in [a,b]$.



Proof: Let $x \in [a,b]$. If $x=a$ then the inequality is trivially satisfied. So suppose $x > a$. Now we apply the MVT to the interval $[a,x]$.

By MVT there is a point $c \in (a,x)$ satisfying

$$f'(c) = \frac{f(x) - f(a)}{x-a}.$$

By assumption, we have $m \leq f'(c) \leq M$ and thus

$$m \le \frac{f(x)-f(a)}{x-a} \le M$$

which implies

$$f(a) + m(x-a) \le f(x) \le f(a) + M(x-a)$$

since $x-a>0$. $\square$

Moreover, if we know that $m < f'(x) < M$ for all $x \in (a,b)$ then we get that

$$f(a)+m(x-a) < f(x) < f(a) + M(x-a) \quad \text{for all } x\in(a,b)$$

(i.e. we get <u>strict</u> inequalities).

<u>Example</u> Use the bounded derivative theorem to show that

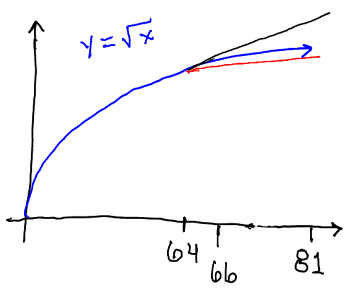
$$8+\tfrac{1}{9} < \sqrt{66} < 8+\tfrac{1}{8}.$$

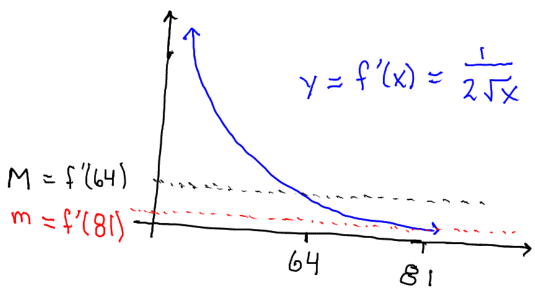
(note that 66 is slightly above 64, so we should expect $\sqrt{66}$ to be slightly bigger than $\sqrt{64}=8$).

<u>Proof</u>: Define f as $f(x)=\sqrt{x}$ so that $f'(x) = \frac{1}{2\sqrt{x}}$ for all $x>0$.

Now f is continuous on $[64,81]$ and differentiable on $(64,81)$.

Moreover, f' is strictly decreasing on this interval.





So we can choose $M=f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{2\cdot 8} = \frac{1}{16}$

and $m = f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{2\cdot 9} = \frac{1}{18}$

so that $m < f'(x) < M$ for all $x \in (64,81)$.

By the Bounded Derivative Theorem,

$$\sqrt{64} + \frac{1}{18}(x-64) < \sqrt{x} < \sqrt{64} + \frac{1}{16}(x-64)$$

for all $x\in(64,81)$. Thus

$$8 + \frac{1}{18}(2) < \sqrt{66} < 8 + \frac{1}{16}(2)$$

and therefore

$$8 + \frac{1}{9} < \sqrt{66} < 8 + \frac{1}{8}. \qquad \square$$

<u>Example</u> If $f(12) = 2$ and $1 \le f'(x) \le 3$ holds for every $x \in \mathbb{R}$, what is the possible range for $f(20)$?

<u>Ans</u>: By BDT,

$$f(12) + 1\cdot(20-12) \le f(20) \le f(12) + 3(20-12)$$

$$\Rightarrow \quad 2 + 8 \le f(20) \le 2 + 3\cdot 8$$

$$\Rightarrow \quad 10 \le f(20) \le 26.$$

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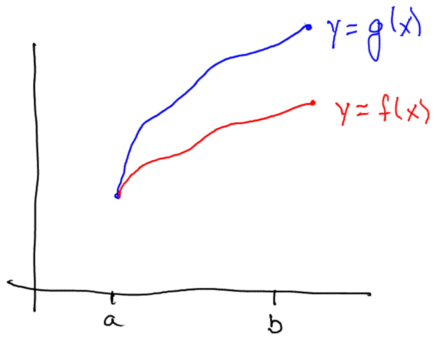
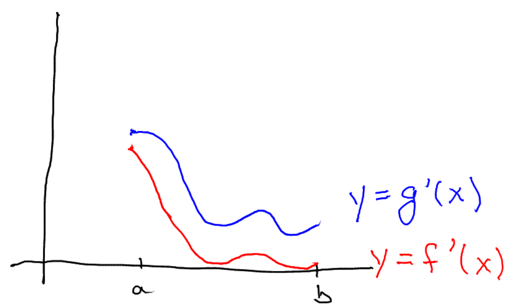
§ 4.2.4 Comparing Derivatives

<u>Theorem (4.8)</u> Assume f and g are continuous on $[a,b]$ and differentiable on (a,b), and assume $f(a) = f(b)$.

1) If $f'(x) \le g'(x)$ for every $x \in (a,b)$
   then $f(x) \le g(x)$ for every $x \in (a,b)$.

2) If $f'(x) \ge g'(x)$ for every $x \in (a,b)$
   then $f(x) \ge g(x)$ for every $x \in (a,b)$.





"If g is always growing faster than f, then g will always be greater than f."

Proof: We will prove (1), since (2) is analogous.

Assume $f'(x) \leq g'(x)$ for every $x \in (a,b)$.

Define h on $[a,b]$ as

$$h(x) = g(x) - f(x)$$

such that $h'(x) = g'(x) - f'(x)$ for all $x \in (a,b)$.

Then $h'(x) \geq 0$ for all $x \in (a,b)$.

Let $x \in (a,b]$ and apply MVT to $[a,x]$. By MVT, there is a point $c \in (a,x)$ so that

$$h'(c) = \frac{h(x) - h(a)}{x-a}.$$

But $h'(c) \geq 0$ and $h(a) = g(a) - f(a) = 0$.

Thus $0 \leq \dfrac{h(x)}{x-a}$ and therefore $0 \leq h(x)$ since $x - a > 0$.

This implies $0 \leq g(x) - f(x)$ and thus $f(x) \leq g(x)$. $\square$

Note: If we instead assume $f'(x) < g'(x)$ for every $x \in (a,b)$ then we get $f(x) < g(x)$ for every $x \in (a,b]$.

Ex Prove that $x \quad \frac{1}{2}x^2 < \ln(1+x) < x$ for every $x > 0$.

Proof: Define functions f, g, and h as

$$f(x) = x - \tfrac{1}{2}x^2$$
$$g(x) = \ln(1+x)$$
$$h(x) = x$$

for every $x \geq 0$. Then $f(0) = g(0) = h(0) = 0$ and

$$f'(x) = 1 - x$$
$$g'(x) = \frac{1}{1+x}$$
$$h'(x) = 1$$

for every $x > 0$.

Now, if $x > 0$, we have $(1+x)(1-x) = 1 - x^2 < 1$

and thus $1 - x < \dfrac{1}{1+x}$ since $1 + x > 0$.

Also, $1 + x > 1$ and thus $\dfrac{1}{1+x} < 1$ for every $x > 0$.

Therefore: $1-x < \frac{1}{1+x} < 1$ for every $x>0$.

Hence $f'(x) < g'(x) < h'(x)$ for every $x>0$.

By Theorem 4.8, we have that $f(x) < g(x) < h(x)$ for every $x>0$.

That is, $x-\frac{1}{2}x^2 < \ln(1+x) < x$ for every $x>0$ $\square$.

We may divide each term of the above inequality by x to find that

$$\frac{x-\frac{1}{2}x^2}{x} < \frac{\ln(1+x)}{x} < 1$$

and thus $1-\frac{1}{2}x < \ln\left[(1+x)^{1/x}\right] < 1$ for every $x>0$

Since the function F defined as $F(y)=e^y$ for all $y\in\mathbb{R}$ is strictly increasing, we get

$$\boxed{e^{1-\frac{1}{2}x} < (1+x)^{1/x} < e^1} \quad \text{for all } x>0.$$

<u>Problem</u>: Prove that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

<u>Proof</u>: For each $n\in\mathbb{N}$, we have $\frac{1}{n}>0$ and thus by the above inequality

$$e^{1-\frac{1}{2n}} < \left(1+\frac{1}{n}\right)^n < e$$

for each $n\in\mathbb{N}$. By the squeeze theorem for sequences,

$$\lim_{n\to\infty} e^{1-\frac{1}{2n}} \le \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le e.$$

But $\lim_{n\to\infty} e^{1-\frac{1}{2n}} = e^{\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)} = e^{1-0} = e$

(since $F(y)=e^y$ is continuous and $\lim_{n\to\infty}\frac{1}{2n}=0$).

Thus $e \le \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le e$. $\square$

More generally, we can prove:

Theorem (4.9) For every $\alpha \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(1+\frac{\alpha}{n}\right)^n = e^\alpha.$$

Proof: exercise.

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§4.3 L'Hôpital's Rule

When computing limits of complicated expressions consisting of continuous functions, e.g. $\lim_{x\to a} \frac{f(x)}{g(x)}$ or $\lim_{x\to a} f(x)^{g(x)}$

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist we can just plug them in!

We must be careful, however, if the limits are zero or infinity. If we get an <u>indeterminate form</u>:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 1^\infty \quad \text{or} \quad 0^0$$

we must do more work.

Theorem (L'Hôpital's Rule – first form)

Let f and g be functions and let $a \in \mathbb{R}$ such that

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ ⇐ That is $\frac{f(x)}{g(x)}$ tends to the indeterminate form $\frac{0}{0}$ as $x \to a$
- there is an open interval I containing a so that f and g are differentiable everywhere on I (except possibly at a)
- $\lim_{x\to a} g'(0) \neq 0$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if this limit exists.

A complete proof of L'hôpital's rule requires a complicated application of the Mean Value Theorem and is beyond the scope of this course.

The main idea, however, is the following:

- We can suppose that $f(a) = 0$ and $g(a) = 0$.

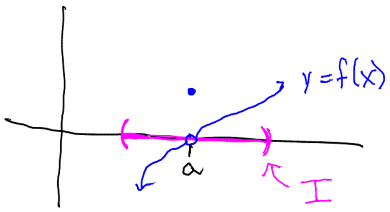
Reasoning

Since f and g are differentiable on I (except possibly at a), they must be continuous on $I \setminus \{a\}$. Now

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

imply that f and g either have $f(a) = 0$ and $g(a) = 0$ or they have possible removeable discontinuities there.

We may therefore suppose $f(a) = 0 = g(a)$ since this does not change the limits).



- Suppose now that $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in I$ with $x \neq a$.

Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \qquad \text{since } g(a) = f(a) = 0$$

$$= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \, \frac{x-a}{x-a}$$

$$= \lim_{x\to a} \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$= \frac{\left[\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}\right]}{\left[\lim\limits_{x\to a} \frac{g(x)-g(a)}{x-a}\right]} = \frac{f'(a)}{g'(a)}.$$

Since f' and g' are continuous, we may assume $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x)$.

$$\text{So} \quad \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \qquad \square$$

Example:

$$\lim_{x \to 0} \frac{\tan x}{x} \qquad \tan 0 = 0 \quad \text{so apply L'Hôpital's Rule}$$

$$= \lim_{x \to 0} \frac{\tan'(x)}{1} = \lim_{x \to 0} \sec^2(x) = \sec^2(0) = \frac{1}{\cos^2(0)} = 1.$$

We can repeatedly apply L'Hôpital's Rule if $f'(a) = 0 = g'(a)$.

Example: $\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^4}$

Let $f(x) = 1 - \cos(x^2)$

$g(x) = x^4$

then $f'(x) = 2x \sin(x^2)$

$g'(x) = 4x^3$

$$\text{so} \quad \lim_{x \to 0} \frac{1 - \cos(x^2)}{x^4} \overset{\text{by L'hopital's}}{=} \lim_{x \to 0} \frac{2x(\sin x^2)}{4x^3}$$

$$= \lim_{x \to 0} \frac{1}{2} \frac{\sin x^2}{x^2}$$

by L'hopital's

$$= \lim_{x \to 0} \frac{1}{2} \frac{2x \cos x^2}{2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \cos x^2 = -1$$