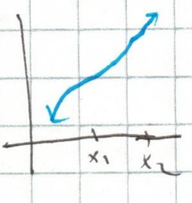
§ 4.2.2

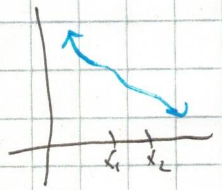
Theorem (Increasing & Decreasing Functions) 4.6

Let f be a diff'ble function on an interval I, and let $x_1, x_2 \in I$ with $x_1 < x_2$.



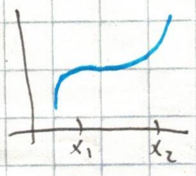
1) If $f'(x) > 0$ for every $x \in I$ then $f(x_1) < f(x_2)$.

That is, strictly f is increasing on I



2) If $f'(x) < 0$ for every $x \in I$ then $f(x_1) > f(x_2)$

That is f is strictly decreasing on I



3) If $f'(x) \geq 0$ for every $x \in I$ then $f(x_1) \leq f(x_2)$

non-decreasing (i.e. ~~not stric~~ increasing but not strictly

4) If $f'(x) \leq 0$ for every $x \in I$ then $f(x_1) \geq f(x_2)$

non-increasing i.e decr. but not strictly

Proof We will prove (1) (The rest are similar)

Suppose $f'(x) > 0$ for every $x \in I$.
Since f is diff'ble on ~~I~~ we may apply MVT to the interval $[x_1, x_2]$

So there is $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f'(c)(x_2 - x_1) = f(x_2) - f(x_1)$$

Now $f'(c) > 0$ and $x_2 - x_1 > 0$

so $f'(c)(x_2 - x_1) > 0$

and thus $0 < f(x_2) - f(x_1)$

or $f(x_1) < f(x_2)$. $\square$

Recall MVT
If f diff'ble on (a,b) and cts on $[a,b]$, there exists $c \in (a,b)$ s.t.
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Q: Is the converse true?

That is: If f is strictly increasing on I is it necessarily the case that $f'(x) > 0$ for every $x \in I$?

A: No! Consider $f(x) = x^3$.

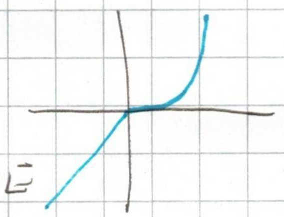
~~This fact~~ The function f is strictly increasing since $f(x_1) < f(x_2)$ for every $x_1, x_2 \in \mathbb{R}$ w/ $x_1 < x_2$.

But $f'(0) = 0$

But that is the only point where f' is zero.

It may also be that f' doesn't exist.

e.g. $f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$



strictly increas everywhere but $f'(0)$ DNE

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§ 4.2.3 Functions w/ bounded derivs

Q: If a car travelling down a road never exceeds 100km/hr, what is the ~~farthest~~ distance the car could have travelled in 1hr? Ans: 100km.

More generally: If $f'(x)$ is bounded on an interval I (i.e. there are $m, M \in \mathbb{R}$ s.t. $m \leq f'(x) \leq M$ holds for every $x \in I$) then ~~[illegible]~~ the values of f are also bounded!

Thm Assume f is cts on $[a,b]$ and diff'ble on (a,b). and that $m, M \in \mathbb{R}$ exist and suppose that

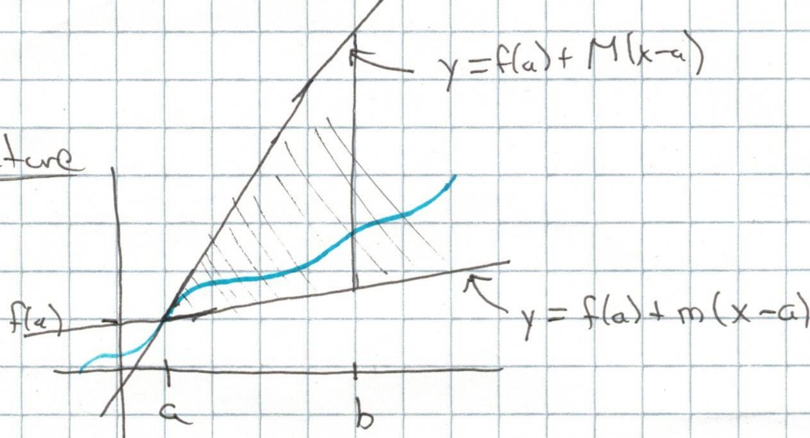
$$m \leq f'(x) \leq M$$

for every $x \in (a,b)$.

Then $$f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a)$$

for every $x \in [a,b]$.

Picture



Proof Let $x \in (a,b]$ and apply MVT to interval $[a,x]$.

There is a point $c \in (a,x)$ s.t.

$$f'(c) = \frac{f(x)-f(a)}{x-a}$$

and thus $\quad f(x) = f(a) + f'(c)(x-a)$.

Since $m \le f'(c) \le M$ we get

$$f(a) + m(x-a) \le f(x) \le f(a) + M(x-a) \quad \square$$

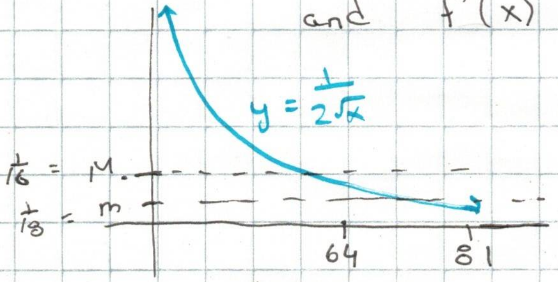
Can be used to estimate:

Ex Prove that $\sqrt{66}$ is between $8+\frac{1}{9}$ and $8+\frac{1}{8}$.

Proof Define f as $f(x)=\sqrt{x}$ so that $f'(x)=\frac{1}{2\sqrt{x}}$ for every $x>0$.

Now f is cont's on $[64,81]$ and diff'ble on $(64,81)$.

and $f'(x) \le f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{2\cdot 8} = \frac{1}{16}$



for every $x \in [64,81)$.

and $f'(x) \ge f'(81) = \frac{1}{2\cdot 9} = \frac{1}{18}$

~~At $x=66$,~~

$$\frac{1}{18} \le f(x) \le \frac{1}{16}$$

$\Downarrow$

$$\sqrt{64} + \frac{1}{18}(66-64) \le \sqrt{66} \le \sqrt{64} + \frac{1}{16}(66-64)$$

$$\Rightarrow \quad 8 + \frac{2}{18} \le \sqrt{66} \le 8 + \frac{2}{16} \quad \Rightarrow \quad \boxed{8+\frac{1}{9} \le \sqrt{66} \le 8+\frac{1}{8}} \quad \square$$

Ex! If $f(12)=2$ and $1 \le f'(x) \le 3$ for every $x \in \mathbb{R}$, what is the possible range for $f(20)$?

$$f(12) + 1(20-12) \le f(20) \le f(12) + 3(20-12)$$

$$\Rightarrow \quad 2 + 1\cdot 8 \le f(20) \le 2 + 3\cdot 8$$

$$\Rightarrow \quad \boxed{10 \le f(20) \le 26}.$$

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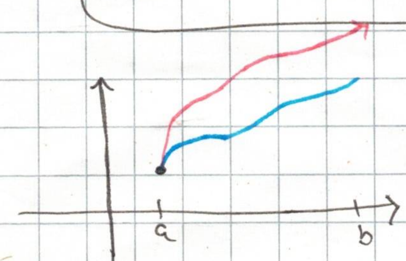
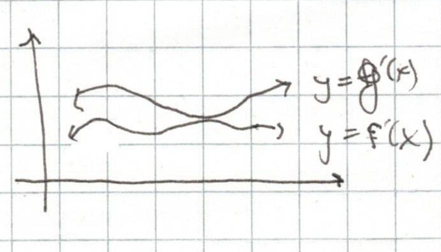
§ 4.2.4 Comparing derivatives

Thm (4.8) Assume f and g are cts on $[a,b]$ and diff'ble on (a,b). and assume $f(a)=g(a)$.

1) If $f'(x) \le g'(x)$ for every $x \in (a,b)$ then $f(x) \le g(x)$ for every $x \in [a,b]$.

2) " " $\ge$ " " " " $\ge$ ...





Pf: We will prove (1) (since (2) is analogous).
Assume $f'(x) \le g'(x)$ for every $x \in (a,b)$.
Define h on $[a,b]$ as
$$h(x) = g(x) - f(x)$$
for every $x \in [a,b]$. s.t. $h'(x) \ge 0$ for every $x \in (a,b)$
~~By MVT, there~~
Let $x \in (a,b]$.
Then h is cts on $[a,x]$ and diff'ble on (a,x).
By MVT, there is $c \in (a,x)$ s.t.

$$h'(c) = \frac{h(x)-h(a)}{x-a}$$

But $h'(c) \ge 0$ and $h(a)=0$

so $h(x) \ge 0$ or $g(x) \ge f(x)$. $\square$.

Note: If we instead assume $f'(x) < g'(x)$ for every $x \in (a,b)$ then we get $f(x) < g(x)$ for every $x \in (a,b]$.

Ex Prove that $x - \frac{1}{2}x^2 < \ln(1+x) < x$ for every $x > 0$.

Pf: ~~Let f(x)~~
Define f, g, and h as

$$f(x) = x - \tfrac{1}{2}x^3 \qquad g(x) = \ln(x+1) \qquad h(x) = x \qquad \text{for every } x \geq 0.$$

Then $f(0) = g(0) = h(0)$

and $f'(x) = 1 - x \quad g'(x) = \frac{1}{1+x} \quad h'(x) = 1$ for every $x > 0$.

~~Hence~~ Now $(1-x)(1+x) = 1 - x^2 < 1$ for every $x > 0$.

so $1 - x < \frac{1}{1+x}$ for every $x > 0$.

Also $1 + x > 1$ for every $x > 0$

so $\frac{1}{1+x} < 1$ for every $x > 0$.

Thus: $1 - x \leq \frac{1}{1+x} < 1$ for every $x > 0$.

Hence $f'(x) < g'(x) < h'(x)$ for every $x > 0$.

By Thm 4.8, $f(x) < g(x) < h(x)$ for every $x > 0$. $\square$

~~Note: the~~ Now: For all $x > 0$, $1 - \frac{1}{2}x \leq \frac{\ln(1+x)}{x} \leq 1$

For all $n \in \mathbb{N}$, $\frac{1}{n} > 0$ so

$$1 - \frac{1}{2n} \leq \frac{\ln(1+\frac{1}{n})}{1/n} \leq 1$$

$$\Rightarrow 1 - \frac{1}{2n} \leq \ln\left(\left(1+\tfrac{1}{n}\right)^n\right) \leq 1$$

$$\Rightarrow e^{1-\frac{1}{2n}} \leq \left(1+\tfrac{1}{n}\right)^n \leq e$$

Exercise:

~~Now, if $n \in \mathbb{N}$ for $x = \frac{1}{n}$ we get~~

$$\lim_{n\to\infty} e^{1-\frac{1}{2n}} = e^{1-\lim_{n\to\infty}\frac{1}{2n}} \quad \text{since} \quad k(x) = e^{x} \text{ is cts.}$$

so $$e \le \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n \le e$$

- More generally:

Thm 4.9 For every $x \in \mathbb{R}$,

$$\boxed{e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}$$