Week 12 (Last Week!)

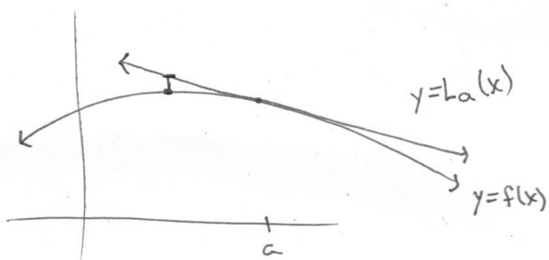
Suppose f is diff'ble at a

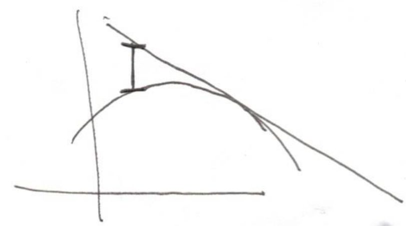
Recall the linear approx. of f at a is the func. L_a^f defined as

$$L_a(x) = f(a) + f'(a)(x-a)$$ ← a polynomal of degree 1

for all $x \in \mathbb{R}$.

Geometrically, the graph of L_a is the line tangent to the graph of f at $(a, f(a))$





$L_a(x)$ "approximates" $f(x)$ for values of x "close" to a.

The error of the approximation:

$$\text{Error}(x) = |f(x) - L_a(x)|.$$

Two things affect how big the error is:

- distance between x and a (i.e. $|\cancel{x}-a|$)
- how "curved" the graph is near a (i.e. $|f''(\alpha)|$)

~~What if we try~~ How might we to get a <u>better</u> approximation of f at a that takes curvature into account?

~~Consider $f(x) = \cos(x)$.~~

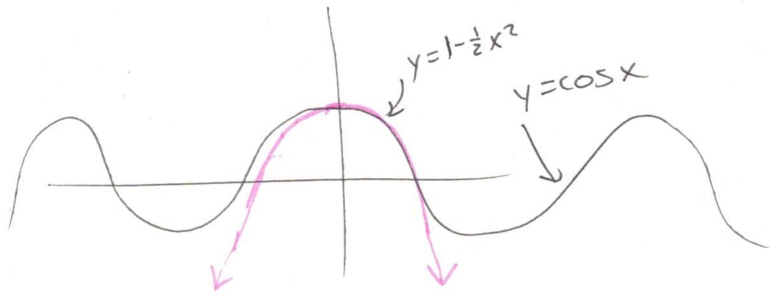
Suppose we want to ~~[illegible]~~ approximate $f(x)$ near zero for x where f is the func defined as

$$f(x) = \cos(x).$$

with higher order polynomials?

We can try making a polynomial function p where

$$p(x) = a + bx + cx^2$$



$$\cos(0) = 1$$
$$\cos'(0) = -\sin(0) = 0$$
$$\cos''(0) = -\cos(0) = -1$$

$$p'(x) = b + 2cx$$
$$p''(x) = 2c$$

should have

$$p(0) = f(0) \qquad p(0) = a \qquad \Rightarrow \quad a = 1$$
$$p'(0) = f'(0) \qquad p'(0) = b \qquad b = 0$$
$$p''(0) = f''(0) \qquad p''(0) = 2c \qquad c = -\frac{1}{2}$$

$$p(0.01) = 1 - \frac{(0.01)^2}{2} = 0.995$$
$$\cos(0.01) = 0.9950042\ldots$$

P. goo

so choose $p(x) = 1 - \frac{1}{2}x^2$

Can get even better approximations with higher derivatives!

Def Suppose a function f is n-times diff'ble at a.
The <u>n^{th}-degree Taylor Polynomial of f centred at a</u> is $T^f_{n,a}$ defined as

$$T^f_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$$

$$= \boxed{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

we write $T_{n,a}$ for simplicity if f is known

This is the unique polynomial with degree n that satisfies

$$p(a) = f(a)$$
$$p'(a) = f'(a)$$
$$\vdots$$
$$p^{(n)}(a) = f^{(n)}(a)$$

Where does this come from?

s'pose $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$

$p'(x) = c_1 + 2c_2(x-a) + \dots + nc_n(x-a)^{n-1}$

$p(a) = c_0$

$p''(x) = 2c_2 + 3\cdot2c_3(x-a) + \dots + n(n-1)c_n(x-a)^{n-2}$

$p'(x) =$

$p'''(x) = 6c_3 + 4\cdot3\cdot2(x-a) + \dots$

$\Rightarrow$ $p(a) = c_0 = \quad = f(a)$

$p'(a) = c_1 = 1c_1 = f'(a)$

$p''(a) = 2c_2 = 2!c_2 = f''(a)$

$p'''(a) = 6c_3 = 3!c_3 = f'''(a)$

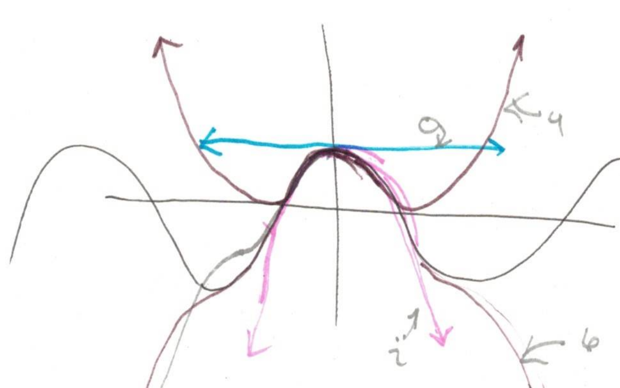
$p^{(k)}(a) = k! = k!c_k =$

$p^{(n)}(a) \quad = n!c_n$

$\Rightarrow$ $\boxed{c_k = \frac{f^{(k)}(a)}{k!}}$

Ex: Find the n^{th} degree Taylor poly's for cos at zero

$f(x) = \cos x$	$f(0) = 1$	$T_{0,0}(x) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$	$T_{1,0}(x) = 1$
$f''(x) = -\cos x$	$f''(0) = -1$	$T_{2,0}(x) = 1 - \frac{1}{2}x^2$
$f'''(x) = \sin x$	$f'''(0) = 0$	$T_{3,0}(x) = 1 - \frac{1}{2}x^2$
$f^{(4)}(x) = \cos x$	$f^{(4)}(0) = 1$	$T_{4,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$
$f^{(5)}(x) = -\sin x$	$f^{(5)}(0) = 0$	$T_{5,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$
$f^{(6)}(x) = -\cos x$	$f^{(6)}(0) = -1$	$T_{6,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$

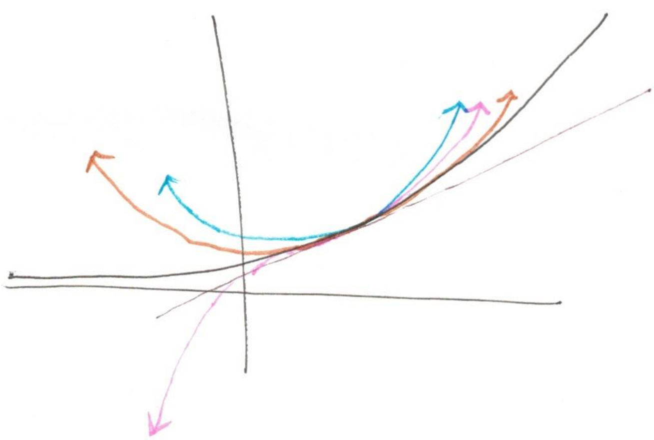


Note: Cos is an even function (i.e. $f(-x) = f(x)$ for all $x \in \mathbb{R}$) so all odd derivs of cos are zero at $x=0$

Ex Find $T_{4,1}(x)$ for $f(x)=e^x$ $a=1$ $n=4$

$$f'(x)=e^x \qquad f(1)=e$$
$$f''(x)=e^x \quad\Rightarrow\quad f'(1)=e$$
$$f'''(x)=e^x \qquad f''(1)=e$$
$$\vdots$$
$$f^{(4)}(1)=e$$

So $T_{4,1}(x) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4$



higher order Taylor ~~approx~~ Poly's give better and better approx's.

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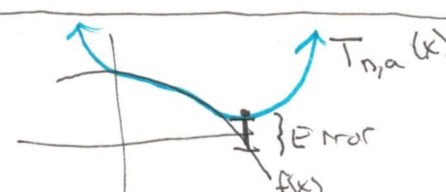
§ 5.2 Taylor's Thm and Errors

Def'n Suppose f is n times diff'ble at a. The <u>nth degree Taylor Remainder function of f centred at a</u> is the function $R_{n,a}$ defined as

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

The error of the Taylor polynomial $T_{n,a}$ at a point x is

$$\text{Error}(x) = |R_{n,a}(x)|.$$



Thm (Taylor's Thm)

Suppose f is n times diff'ble on an interval I containing a.
Let $x \in I$ $x \neq a$.
~~For every $x \in I$~~ There is a point c in I between x and a

s.t.

$$\boxed{R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

Corollary If $M > 0$ is a number such that
$$|f^{(n+1)}(c)| \leq M$$
for every point c between x and a then

$$\text{Error}(x) = |R_{n,a}(x)| \leq \frac{M}{(n+1)!}|(x-a)^{n+1}|$$

- Similar to Bounded Deriv theorem.
- Idea: If $|f^{(n+1)}(x)| \leq M$ for all $x \in I$ then

we get useful bounds on the error:

$$\text{Error}(x) = |f(x) - T_{n,a}(x)| < \frac{M}{(n+1)!}(x-a)^{n+1}.$$

We won't prove Taylor's Thm, but consider the following observations

Notes

1) When $n=1$, $T_{1,a}(x) = f(a) + f'(a)(x-a)$
$= L_a(x)$.

and $|R_{1,a}(x)| = \left|\frac{f''(c)}{2}\right|(x-a)^2$

This is the linear approximation error from earlier.

2) When $n=0$, $T_{0,a}(x) = f(a)$. So Taylor's thm says there is a point c between a and x s.t.

$$f(x) - f(a) = f'(c)(x-a)$$

or $f'(c) = \dfrac{f(x)-f(a)}{x-a}$. This is the MVT!

So Taylor's Thm is higher order generalization of MVT.

3). Theorem doesn't tell us how to find c, but if we can find an upper bd on $|f^{(n+1)}(c)|$ then we get an upper bd on error.

$$\text{Error}(x) \le \frac{M}{(n+1)!} |(x-a)^{n+1}|$$

Ex ~~Consider f defined as $f(x) = \sqrt{1+x}$ for all $x \geq -1$.~~

use 2nd order Taylor poly to approximate $\sqrt{1.1}$ and find an upper bound on the error in the approx.

Define $f(x) = \sqrt{1+x}$

sol • $a = 0$ $\quad f(0) = 1$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2(1+x)^{1/2}} = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4(1+x)^{3/2}}$$

$f(0) = 1$

$f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

So $$T_{2,0}(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)}{2!}x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2}x^2$$

$$\boxed{T_{2,0}(x) = 1 + \frac{x}{2} - \frac{x^2}{8}}$$

• $$T_{2,0}(0.1) = T_{2,0}\left(\frac{1}{10}\right)$$

$$= 1 + \frac{1}{2}\frac{1}{10} - \frac{1}{8}\frac{1}{100}$$

$$= 1 + \frac{1}{20} - \frac{1}{800} = \frac{839}{800}$$

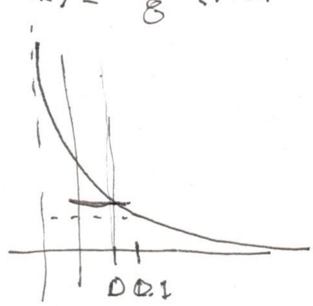
$$= 1 + 0.05 - 0.00125 = 1.04875$$

• $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad$ positive and decreasing on $[1, 1.1]$

So $0 < f'''(c) \le f'''(1) = \frac{3}{8}$

for all $c \in [1, 1.1]$.

So $|f'''(c)| \le \frac{3}{8}$. $\quad$ Choose $M = \frac{3}{8}$



Therefore:

~~Error (x)~~

$$\text{Error}(0.1) = |R_{2,0}(0.1)| \le \frac{\frac{3}{8}|0.1|^3}{3!} \quad \text{by Taylor's Thm.}$$

$$= \frac{3}{8}\frac{1}{2\cdot3}\left(\frac{1}{10}\right)^3$$

$$= \frac{1}{16}\,\frac{1}{1000} = \frac{1}{16000} = 0.0000625$$

~~Q:Is~~

So: $T_{2,0}(0.1) = \frac{839}{800} = 1.04875 \approx \sqrt{1.1}$

Q: Is this an over estimate or an underestimate?

(i.e. is $\frac{839}{800} > \sqrt{1.1}$ or $\frac{839}{800} < \sqrt{1.1}$?)

Well, Taylor's Thm says there is a $c \in (0, 0.1)$ s.t.

$$f(0.1) - T_{2,0}(0.1) = R_{2,0}(0.1) = \frac{f^{(3)}(c)}{3!}(0.1)^3$$

But $0 < f^{(3)}(c) \le f^{(3)}(0) = \frac{3}{8}$ ~~$f^{(3)}(0.1) < f^{(3)}(c) < f^{(3)}(0)$~~

so $f(0.1) - T_{2,0}(0.1) > 0$

$$\Rightarrow \sqrt{1.1} > T_{2,0}(0.1)$$

$$\Rightarrow \sqrt{1.1} > \frac{839}{800}$$

So this is an under estimate.

Thus, ~~our~~ the true value is in the range

$$\cancel{\sqrt{1.1}} < \quad \frac{839}{800} < \sqrt{1.1} < \frac{839}{800} + \frac{1}{16000}$$

$$1.04875 < \sqrt{1.1} < 1.0488125$$

$$\Rightarrow \sqrt{1.1} \approx 1.0488$$