MATH 137 — Fall 2020
Practice Problems

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1. Prove using only the definition of the limit that $\lim_{x\to 4}(x^2 - 3x + 2) = 6$.

Solution. We will show that, for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x - 4| < \delta \implies |(x^2 - 3x + 2) - 6| < \epsilon$.

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min\{1, \frac{\epsilon}{6}\}$. Let $x \in \mathbb{R}$ be given and suppose that $|x-4| < \delta$. Then $|x-4| < 1$ which implies that $3 < x < 5$ and thus $4 < x+1 < 6$. Hence $|x+1| \leq 6$. Now

$$\begin{aligned}
|(x^2 - 3x + 2) - 6| &= |x+1||x-4| \\
&\leq 6|x-4| && \text{since } |x+1| \leq 6 \\
&< \epsilon && \text{since } |x-4| < \frac{\epsilon}{6}
\end{aligned}$$

as desired. □

Here is an alternate proof that chooses δ in a different way.

Alternate proof. Let $\epsilon > 0$ be given and choose $\delta = \min\{\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{10}\}$. Let $x \in \mathbb{R}$ be given and suppose that $|x - 4| < \delta$. Then $|x-4| < \sqrt{\frac{\epsilon}{2}}$ and $|x-4| < \frac{\epsilon}{10}$. Now

$$\begin{aligned}
|(x^2 - 3x + 2) - 6| &= |x-4||x+1| \\
&= |x-4||x-4+5| \\
&\leq |x-4|(|x-4|+5) && \text{(by the Triangle inequality)} \\
&= |x-4|^2 + 5|x-4| \\
&< \left(\sqrt{\frac{\epsilon}{2}}\right)^2 + 5\left(\frac{\epsilon}{10}\right) && \text{(since } |x-4| < \sqrt{\frac{\epsilon}{2}} \text{ and } |x-4| < \frac{\epsilon}{10}\text{)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

as desired. □

2. Suppose $a_1, a_2, a_3 \ldots$ is a sequence that converges to 2, and suppose further that $a_n \neq 5$ for all $n \in \mathbb{N}$. Show (using only the definition of the limit) that $\lim_{n\to\infty} \frac{3}{5 - a_n} = 1$.

Solution. We will show that, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\left|\frac{3}{5-a_n} - 1\right| < \epsilon$ for every $n \geq N$.

Proof. Let $\epsilon > 0$ be given. Define $\epsilon_1 = \min\{1, 2\epsilon\}$ and note that $\epsilon_1 > 0$. Because it is assumed that $\lim_{n\to\infty} a_n = 2$, there is a number $N \in \mathbb{N}$ such that $|a_n - 2| < \epsilon_1$ for every $n \geq N$. Let $n \geq N$ be given. Then $|a_n - 2| < \epsilon_1$ and thus $|a_n - 2| < 1$. It follows that $1 < a_n < 3$ which implies that $-4 < a_n - 5 < -2$ and thus $|a_n - 5| \geq 2$. Now

$$\begin{aligned} \left|\frac{3}{5-a_n} - 1\right| &= \left|\frac{a_n - 2}{a_n - 5}\right| \\ &\leq \frac{|a_n - 2|}{2} && \left(\text{since } |a_n - 5| \geq 2 \text{ and thus } \frac{1}{|a_n - 5|} \leq \frac{1}{2}\right) \\ &< \epsilon && (\text{since } |a_n - 2| < 2\epsilon) \end{aligned}$$

as desired. □

3. Let $L \in \mathbb{R}$ and let f be a function such that $\lim_{x\to\infty} f(x) = L$. For each of the following statements, either prove it is true (using only the definition of the limit) or show it is false by providing a counterexample.

(a) $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$.

Solution. This statement is true. We will show that, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(\frac{1}{x}) - L| < \epsilon$ for every $x \in (0, \delta)$.

Proof. Let $\epsilon > 0$ be given. Because it is assumed that $\lim_{x\to\infty} f(x) = L$, there is a number $N > 0$ such that $|f(x) - L| < \epsilon$ for every $x > N$. Define $\delta = \frac{1}{N}$ such that $\delta > 0$. Let $x \in \mathbb{R}$ be given and suppose that $x \in (0, \delta)$. Then $0 < x < \delta$ and thus $0 < \frac{1}{\delta} < \frac{1}{x}$. This implies that $\frac{1}{x} > N$ and thus

$$\left|f\left(\frac{1}{x}\right) - L\right| < \epsilon,$$

as desired. □

(b) $\lim_{x\to 0} f\left(\frac{1}{x}\right) = L$.

Solution. This statement is FALSE. The idea is that $\frac{1}{x} \to 0^+$ as $x \to \infty$ and thus $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$. However, this does not tell us anything about the left-sided limit $\lim_{x\to 0^-} f\left(\frac{1}{x}\right)$. So we should be able to find a counterexample where $\lim_{x\to 0^-} f\left(\frac{1}{x}\right) \neq \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$.

Proof (that this is false). Define a function f as

$$f(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

such that $\lim_{x\to\infty} f(x) = 1$. From part (a), we also have that the right-sided limit is $\lim_{x\to 0^+} f(\frac{1}{x}) = 1$. However, note that $\frac{1}{x} < 0$ and thus $f(\frac{1}{x}) = -1$ for all $x < 0$. Hence we can conclude that the left-sided limit is equal to $\lim_{x\to 0^-} f(\frac{1}{x}) = -1$. Since the two one-sided limits at $x = 0$ do not agree, it follows that $\lim_{x\to 0} f\left(\frac{1}{x}\right)$ does not exist (and thus not equal to L). □

4. Let f be a function such that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Prove that f is differentiable at $x = 0$ and that $f'(0) = 0$.

Solution. The idea here is to use the Squeeze Theorem and the definition of the derivative to show that $\lim_{h\to 0} \frac{f(h) - f(0)}{h} = 0$.

Proof. First note that $|f(0)| \leq 0$ by assumption and thus $f(0) = 0$. Moreover, for every $h \in \mathbb{R}$ it holds that $-h^2 \leq f(h) \leq h^2$, or equivalently

$$-|h|^2 \leq f(h) \leq |h|^2.$$

Since $|h| > 0$ whenever $h \neq 0$, we may divide the above inequality by $|h|$ to find that

$$-|h| \leq \frac{f(h)}{|h|} \leq |h|$$

and thus

$$-|h| \leq \frac{f(h) - f(0)}{h} \leq |h|$$

for every $h \neq 0$ (where we use the fact that $f(0) = 0$). From the Squeeze Theorem, it follows that

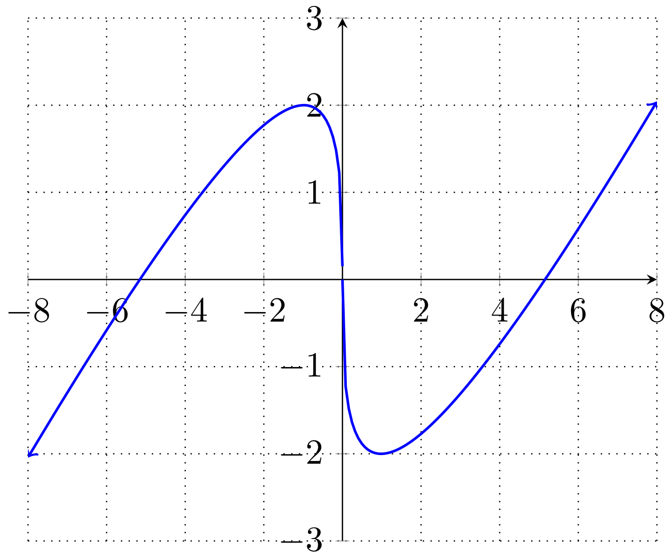
$$\lim_{h\to 0} -|h| \leq \lim_{h\to 0} \frac{f(h) - f(0)}{h} \leq \lim_{h\to 0} |h|.$$

Finally, since $\lim_{h\to 0} -|h| = 0 = \lim_{h\to 0} |h|$ and $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$, we conclude that $f'(0) = 0$. □

5. For each of the following functions, find all of the following: (i) the domain, (ii) the range, (iii) critical points, (iv) intervals where the function is increasing/decreasing, (v) intervals where the function is concave up/down, (vi) inflection points, (vii) local minima/maxima, (viii) global minimum/maximum (if they exist).

(a) $f(x) = x - 3x^{1/3}$

Solution. Let's first plot the graph of this function:



i. This function is defined for all x. Therefore $\text{dom}(f) = \mathbb{R}$.

ii. This function is continuous on all of $\mathbb{R}$. Moreover, we have the limits $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. Thus the range of f is all of $\mathbb{R}$.

iii. The derivative is $f'(x) = 1 - x^{-2/3} = \frac{1}{x^{2/3}}(x^{2/3} - 1)$, which is equal to zero when $x = \pm 1$. The critical points are therefore at $x = \pm 1$ and $x = 0$ (where the derivative doesn't exist).

iv. Note that we may write $x^{2/3}$ as $(x^{1/3})^2$, so we may express the derivative as

$$f'(x) = \frac{1}{x^{2/3}}(x^{1/3} - 1)(x^{1/3} + 1).$$

To see the behaviour of the function, we can construct the following table

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$x^{2/3}$	$+$	$+$	$+$	$+$
$(x^{1/3} - 1)$	$-$	$-$	$-$	$+$
$(x^{1/3} + 1)$	$-$	$+$	$+$	$+$
$f'(x)$	$+$	$-$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Thus f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$, while it is decreasing on $[-1, 1]$.

v. The second derivative is $f''(x) = \frac{2}{3}x^{-5/3}$, which is never equal to zero but does not exist at $x = 0$. Note that $x^{-5/3}$ is negative when x is negative, and is positive when x is positive. This gives us the table

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\frown$	$\smile$

Thus f is concave down on $(-\infty, 0]$ and concave up on $[0, \infty)$.

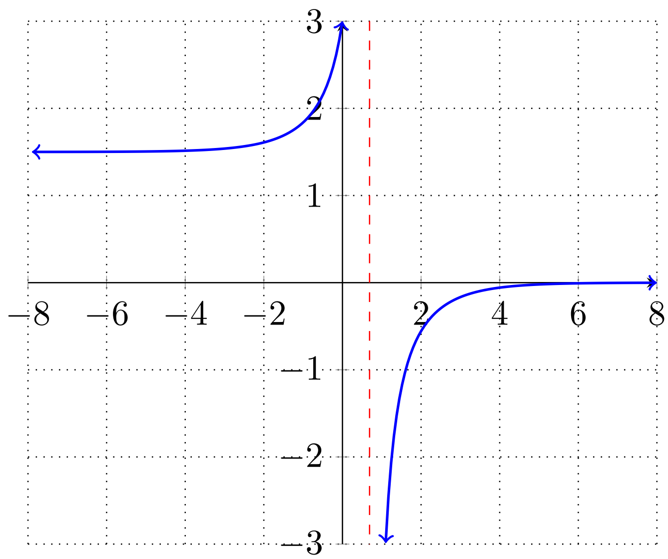
vi. The second derivative is never equal to zero. Thus the graph of f has no inflection points.

vii. There is a local maximum at -1 and a local minimum at 1.

viii. Note that $\lim_{x\to\pm\infty} = \pm\infty$ and thus f has neither a global min nor max.

(b) $f(x) = \frac{3}{2-e^x}$

Solution. Let's first plot the graph of this function:



i. This function is defined for all x except when $2 - e^x = 0$, which occurs when $x = \ln 2$. Therefore $\text{dom}(f) = \{x \in \mathbb{R} : x \neq \ln 2\} = \mathbb{R} \setminus \{\ln 2\}$.

ii. Here we must find all values $y \in \mathbb{R}$ for which $y = f(x)$ has a solution. First note that $f(x) = 0$ has no solutions and thus 0 is not in the range of f. So suppose that $y \neq 0$ and suppose that $y = \frac{3}{2-e^x}$ has a solution. Then

$$e^x = 2 - \frac{3}{y}$$

which has a solution if and only if $2 - \frac{3}{y} > 0$ (since e^x is always positive), or equivalently $\frac{2}{3} > \frac{1}{y}$. There are two cases:

- Suppose $y > 0$. Then y is in the range if and only if $y > \frac{3}{2}$.
- Suppose $y < 0$. Then y is in the range if and only if $y < \frac{3}{2}$. Hence every $y < 0$ is in the range of f.

Hence $\text{range}(f) = \{y \in \mathbb{R} : y < 0 \text{ or } y > \frac{3}{2}\} = (-\infty, 0) \cup (\frac{3}{2}, \infty)$.

iii. The derivative is given by $f'(x) = \frac{3e^x}{(2-e^x)^2}$ which is never equal to zero and does not exist when $x = \ln 2$. But f is not defined at $\ln 2$, so it does not have a critical point there. Thus f has no critical points.

iv. Note that $e^x > 0$ for all x and that $(2 - e^x)^2 > 0$ for all $x \neq \ln 2$. Hence $f'(x) > 0$ everywhere on its domain and thus f is increasing on its entire domain.

v. The second derivative is given by $f''(x) = \frac{3e^x(2+e^x)}{(2-e^x)^3}$. Note that $f''(x) < 0$ when $2 - e^x < 0$ (or equivalently when $x > \ln 2$) and that $f''(x) > 0$ when $2 - e^x > 0$ (or equivalently when $x < \ln 2$). Thus f is concave up on $(-\infty, \ln 2)$ and concave down on $(\ln 2, \infty)$.

vi. The second derivative is never equal to zero. Thus the graph of f has no inflection points.

vii. There are no local extrema as f has no critical points.

viii. Note that $\lim_{x \to \ln 2^{\pm}} f(x) = \mp\infty$ and thus f has neither a global min nor max.

6. Consider the limit $\lim_{x \to \infty} \dfrac{x + \sin x}{x + 1}$.

(a) Explain which conditions are satisfied for applying L'hopital's rule for this limit. What happens when you apply L'hopital's rule?

Solution. First let's show that $\lim_{x \to \infty} (x + \sin x) = \infty$.

Proof. Let $M > 0$ be given. Pick $N = M + 1$ and let x be a real number such that $x \geq N$. (We will show that $x + \sin x > M$.) Now $\sin x \geq -1$ and thus

$$x + \sin x \geq x - 1 > N - 1 = M$$

as desired. □

It is clear that $\lim_{x \to \infty} (x+1) = \infty$ and thus both the numerator and the denominator of the limit $\lim_{x \to \infty} \dfrac{x + \sin x +}{x}$ diverge to infinity. If we apply L'hopital's rule, we find

$$\lim_{x\to\infty} \frac{x+\sin x}{x+1} = \lim_{x\to\infty} \frac{1+cosx}{1} = \lim_{x\to\infty}(1+\cos x).$$

However, this limit doesn't exist! Indeed, consider the sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ defined as

$$a_n = 2n\pi \qquad \text{and} \qquad b_n = (2n+1)\pi$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$, but $(1+\cos a_n) = 2$ and $(1+\cos b_n) = 0$ for all $n \in \mathbb{N}$. So the limit $\lim_{x\to\infty}(1+\cos x)$ doesn't exist.

(b) Use another method to compute this limit.

Solution. We may compute the limit using other methods. Note that

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

for all $x > 0$ and thus $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. Now

$$\lim_{x\to\infty} \frac{x+\sin x}{x+1} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1+\frac{1}{x}} = \frac{1+\lim_{x\to\infty}\frac{\sin x}{x}}{1+\lim_{x\to\infty}\frac{1}{x}} = \frac{1+0}{1+0} = 1.$$

(c) What conditions for applying L'hopital's rule are *NOT* met for this limit? Use this to explain why the answers from (a) and (b) are different.

Solution. Recall what the Theorem regarding L'hopital's rule says:

Theorem. Suppose that $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$ and suppose that at least one of the following are true:

- $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$.
- $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if this limit exists.

The important part is that the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ must **exist** for us to be able to apply L'hopital's rule!