# MATH 137 — Fall 2020
# Solutions to Written Assignments and Tips for Writing Good Proofs

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The solutions provided by the head instructor provide good explanations for the solutions, but don't showcase what a good solution should look like. I've compiled here what "good and complete" solutions to the assignments should look like for you to get an idea of how to write yours.

## Contents

## Tips for writing good proofs

- Clearly state at the beginning what you are trying to prove.
- Start your proof with "*Proof*:" and end with "*QED*" or □.
- Declare your variables before using them.
  - If the statement you are proving has "For all $x$, $P(x)$" then in your proof you should write something like "Let $x$ be given." Then prove $P(x)$ for the arbitrarily given value of $x$.
  - If the statement you are proving has "There exists $x$ such that $P(x)$" then in your proof you should write something like "Let $x$ be (what ever you choose it to be)." Then Prove $P(x)$ for this particular value of $x$.
- Explain your reasoning at each step using words. (You should be able to read your proof aloud in complete sentences.)
- Clearly state what your assumptions are and when you are using them.

## Assignment 1

Let $b$ be a real number. Show that the inequality

$$|x+1| + 2|x-1| < b \tag{1}$$

has no solutions if and only if $b \leq 2$.

### Solution

There essentially two things to prove here. The two statements to be proved are:

- If $b > 2$ then there exists $x \in \mathbb{R}$ such that $|x+1| + 2|x-1| < b$ holds.
- If $b \leq 2$ then for every $x \in \mathbb{R}$ it holds that $|x+1| + 2|x-1| \not< b$.

We will prove these statements below.

**Claim.** *If $b > 2$ then the inequality in* (1) *has a solution.*

*Proof.* Suppose that $b > 2$. Let $x = 1$. Then

$$\begin{aligned} |x+1| + 2|x-1| &= |1+1| + 2|1-1| \\ &= 2 \\ &< b \end{aligned}$$

and thus the inequality in (1) has a solution. □

**Claim.** *If $b \leq 2$ then the inequality in* (1) *does **not** have a solution.*

*Proof.* Suppose that $b \leq 2$ and let $x \in \mathbb{R}$ be given. There are three cases to consider:

Case 1: Suppose that $x < -1$. Then $|x+1| = -(x+1)$ and $|x-1| = 1-x$, and thus

$$\begin{aligned} |x+1| + 2|x-1| &= -(x+1) + 2(1-x) \\ &= 1 - 3x \\ &> 4 \\ &> 2 \\ &\geq b, \end{aligned}$$

where we use the fact that $x < -1$ and thus $-3x > 3$.

Case 2: Suppose that $-1 \leq x < 1$. Then $|x+1| = x+1$ and $|x-1| = 1-x$, and thus

$$\begin{aligned} |x+1| + 2|x-1| &= x + 1 + 2(1-x) \\ &= 3 - x \\ &> 2 \\ &\geq b, \end{aligned}$$

where we use the fact that $x < 1$ and thus $-x > -1$.

Case 3: Suppose that $1 \leq x$. Then $|x+1| = x+1$ and $|x-1| = x-1$, and thus

$$\begin{aligned} |x+1| + 2|x-1| &= x + 1 + 2(x-1) \\ &= 3x - 1 \\ &\geq 2 \\ &\geq b, \end{aligned}$$

where we use the fact that $x \geq 1$ and thus $3x \geq 3$.

In each case, it holds that $|x+1| + 2|x-1| \geq b$ and thus the inequality in (1) has no solutions. □

# Assignment 2

Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_n \geq 0$ for each $n \in \mathbb{N}$, and let $L$ be a number such that

$$\lim_{n\to\infty} a_n = L.$$

(a) Prove that $L \geq 0$.

(b) Prove that $\lim_{n\to\infty} \dfrac{2}{a_n + 5} = \dfrac{2}{L+5}$.

**Solution.**

**Part (a)**

**Claim.** *It holds that $L \geq 0$.*

*Proof.* Towards a contradiction, assume that $L < 0$ and define $\epsilon = -\frac{L}{2}$ (which is positive). As it is assumed that $\lim_{n\to\infty} a_n = L$, there exists a number $N$ such that $|a_n - L| < \epsilon$ for every $n \geq N$. Hence for every $n \geq N$ we have that

$$\frac{L}{2} < a_n - L < -\frac{L}{2}$$

which implies that $a_n < \frac{L}{2}$ and thus $a_n < 0$. This contradicts the assumption that $a_n \geq 0$. We conclude that $L \geq 0$. □

**Part (b)**

**Claim.** $\lim_{n\to\infty} \dfrac{2}{a_n + 5} = \dfrac{2}{L+5}$.

We will prove that: For all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\left|\frac{5}{a_n+2} - \frac{5}{L+2}\right| < \epsilon$ for all $n \geq N$.

*Proof.* Let $\epsilon > 0$ be given and define $\epsilon_1 = \frac{2\epsilon(L+2)}{5}$. Note that $\epsilon_1 > 0$ as $L \geq 0$. As it is assumed that the sequence $a_1, a_2, \ldots$ converges to $L$, there exists a number $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon_1$ for every $n \geq N$. Let $n \in \mathbb{N}$ and suppose that $n \geq N$. Note that

$$\begin{aligned} |(a_n + 2)(L + 2)| &= (a_n + 2)(L + 2) \\ &\geq 2(L + 2) \\ &> 0 \end{aligned}$$

since both $a_n \geq 0$ and $L \geq 0$, and thus

$$0 < \frac{1}{|(a_n + 2)(L + 2)|} \leq \frac{1}{2(L + 2)}. \tag{2}$$

Now

$$\begin{aligned}
\left|\frac{5}{a_n+2} - \frac{5}{L+2}\right| &= 5\left|\frac{L-a_n}{(a_n+2)(L+2)}\right| \\
&= 5\frac{|a_n-L|}{|(a_n+2)(L+2)|} \\
&\leq \frac{5}{2(L+2)}|a_n-L| && \text{(from (2))} \\
&< \frac{5}{2(L+2)}\epsilon_1 && \text{(since } |a_n-L|<\epsilon_1\text{)} \\
&= \epsilon, && \text{(since } \epsilon_1 = \tfrac{2\epsilon(L+2)}{5}\text{)}
\end{aligned}$$

as desired. □

## Assignment 3

Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 1$ and $a_{n+1} = 4 - \frac{2}{a_n}$ for every $n \in \mathbb{N}$. Show that the sequence converges and compute its limit.

### Solution

We will make use of the Monotone Convergence Theorem to prove that the sequence converges. To do so, we must first show that the sequence is non-decreasing and bounded above. We will prove the following claim.

**Claim.** *For every $n \in \mathbb{N}$ it holds that $1 \leq a_n \leq a_{n+1} \leq 4$.*

*Proof.* We proceed by induction.

(Base) Note that $a_1 = 1$ and $a_2 = 2$ and thus the statement is true when $n = 1$.

(IH) Let $k \in \mathbb{N}$ and suppose that $1 \leq a_k \leq a_{k+1} \leq 4$.

(IS) Note that

$$\begin{aligned}
1 \leq a_k \leq a_{k+1} \leq 4 \quad &\Longrightarrow \quad 1 \geq \frac{1}{a_k} \geq \frac{1}{a_{k+1}} \geq \frac{1}{4} \\
&\Longrightarrow \quad -2 \leq -\frac{2}{a_k} \leq -\frac{2}{a_{k+1}} \leq -\frac{1}{2} \\
&\Longrightarrow \quad 2 \leq 4 - \frac{2}{a_k} \leq 4 - \frac{2}{a_{k+1}} \leq 4 - \frac{1}{2} \\
&\Longrightarrow \quad 2 \leq a_{k+1} \leq a_{k+2} \leq \frac{7}{2} \\
&\Longrightarrow \quad 1 \leq a_{k+1} \leq a_{k+2} \leq 4.
\end{aligned}$$

From the Principle of Mathematical Induction, we conclude that $1 \leq a_n \leq a_{n+1} \leq 4$ holds for every $n \in \mathbb{N}$. □

We will now show that the sequence converges to $2 + \sqrt{2}$.

**Claim.** $\lim_{n\to\infty} a_n = 2 + \sqrt{2}$.

*Proof.* From the previous claim, we conclude that the sequence is non-decreasing and bounded above. It follows from the Monotone Convergence Theorem that the sequence converges. That is, there is a number $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$. Moreover the limit $L$ is the least upper bound of the sequence. Therefore $a_1 \leq L$ (that is, $1 \leq L$) and thus $L \neq 0$. Now

$$\begin{aligned}
L = \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\
&= \lim_{n\to\infty} \left(4 - \frac{2}{a_n}\right) \\
&= 4 - \frac{2}{L}
\end{aligned}$$

and thus $L$ must satisfy $L^2 - 4L + 2 = 0$. It follows that either $L = 2 + \sqrt{2}$ or $L = 2 - \sqrt{2}$. Observe that $a_1 = 1 > 2 - \sqrt{2}$ and thus $2 - \sqrt{2}$ is not an upper bound of the sequence. We conclude that $L = 2 + \sqrt{2}$. $\square$

## Assignment 4

Prove that $\lim_{x \to 2}(x^3 - 7x + 1) = -5$.

### Solution.

We will prove the following statement: For every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in \mathbb{R}$, $0 < |x - 2| < \delta \implies |x^3 - 7x + 1 - (-5)| < \epsilon$.

*Proof.* Let $\epsilon > 0$ be given and define $\delta = \min\{1, \frac{\epsilon}{12}\}$. Let $x \in \mathbb{R}$ and suppose that $0 < |x - 2| < \delta$. As $|x - 2| < 1$, it holds that $-1 < x - 2 < 1$ and thus $1 < x < 3$. Hence

$$0 < x - 1 < 2 \qquad \text{and} \qquad 4 < x + 3 < 6,$$

and thus $0 < |x - 1| < 2$ and $0 < |x + 3| < 6$. Therefore

$$0 < |x - 1||x + 3| < 12. \tag{3}$$

Moreover, it also holds that $|x - 2| < \frac{\epsilon}{12}$. Now

$$\begin{aligned} |x^3 - 7x + 1 - (-5)| &= |x^3 - 7x + 6| \\ &= |x - 1||x + 3||x - 2| \\ &< 12\frac{\epsilon}{12} \\ &= \epsilon, \end{aligned}$$

and thus $|x^3 - 7x + 1 - (-5)| < \epsilon$, as desired. □

## Assignment 5

Suppose $c$ and $d$ are real numbers (with $c \neq 0$) and consider the function $f$ defined as

$$f(x) = \begin{cases} cx + 4d & x < 2 \\ x^2 + 4 & 2 \leq x \leq 3 \\ dx^2 + \frac{2x}{c} + 1 & x > 3 \end{cases}$$

for every $x \in \mathbb{R}$. For which values of $c$ an $d$ is $f$ continuous everywhere?

### Solution.

The function $f$ is continuous everywhere if and only if either $c = 2$ and $d = 1$, or $c = -\frac{2}{3}$ and $d = \frac{7}{3}$. A plot of the graphs for both cases are shown below.

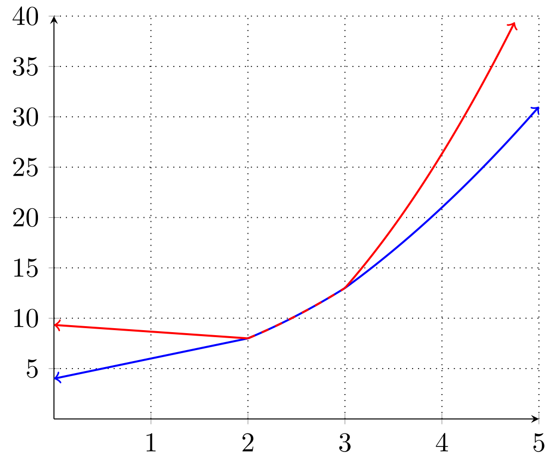



Figure 1: A plot of the graphs of $f$ for different values of $c$ and $d$. The blue curve is the graph when $c = 2$ and $d = 1$. The red curve is the graph when $c = -\frac{2}{3}$ and $d = \frac{7}{3}$.

*Proof.* First note that the function $f$ is clearly continuous on $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$. It remains to find the conditions under which $f$ is continuous at $x = 2$ and $x = 3$. That is, $f$ is continuous everywhere if and only if

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x) \qquad \text{and} \qquad \lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x).$$

Note that $f(2) = 8$ and $f(3) = 13$ and the left- and right-sided limits of $f$ at $x = 2$ and $x = 3$ are

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} \big(cx + 4d\big) = 2c + 4d \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} \big(x^2 + 4\big) = 8 = f(2) \\ \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} \big(x^2 + 4\big) = 13 = f(3) \\ \text{and} \qquad \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} \big(dx^2 + \tfrac{2x}{c} + 1\big) = 9d - \tfrac{6}{c} + 1. \end{aligned}$$

Therefore, the function $f$ is continuous everywhere if and only if

$$2c + 4d = 8 \qquad \text{and} \qquad 9d - \frac{6}{c} + 1 = 13. \tag{4}$$

Since $c \neq 0$ we may multiply the second equation by $c$ to see that

$$\begin{aligned} 9d + \frac{6}{c} + 1 = 13 \qquad &\iff \qquad 9cd + 6 - 12c = 0 \\ &\iff \qquad c(3d-4) + 2 = 0. \end{aligned}$$

The equations in (4) are therefore satisfied if and only if

$$c = 4 - 2d \qquad \text{and} \qquad c(3d-4) + 2 = 0. \tag{5}$$

Plugging $c = 4 - 2d$ into the left-hand side of the second equation, we have

$$(4-2d)(3d-4) + 2 = -6d^2 + 20d - 14 = -2(3d^2 - 10 + 7) = -2(3d-7)(d-1).$$

and thus $(4-2d)(3d-4)+2 = 0$ if and only if either $d = 1$ or $d = \frac{7}{3}$. Plugging these values into the first equation of (5), we see that $f$ is continuous everywhere if and only if either $d = 1$ and $c = 2$ or $d = \frac{7}{3}$ and $c = -\frac{2}{3}$. □

## Assignment 6

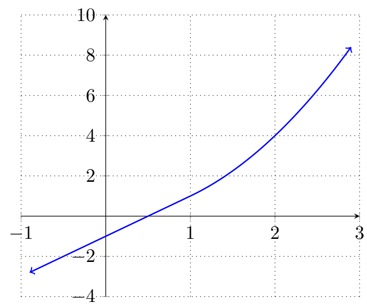
Suppose $a$ and $b$ are real numbers and consider the function $f$ defined as

$$f(x) = \begin{cases} 2x + b & x < a \\ x^2 & x \geq a \end{cases}$$

for every $x \in \mathbb{R}$. For which values of $a$ an $b$ is $f$ differentiable at $a$?

### Solution.

The function $f$ is differentiable at $a$ if and only if $a = 1$ and $b = -1$. The graph of the function with these values is shown below. (Note how the graph smoothly transistions at $x = 1$.)




*Proof.* Note that $f$ is differentiable at $a$ if and only if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. This limit exists if and only if the two one-sided limits

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

exist and their values are the same. Now

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0^-} \frac{2(a+h) + b - a^2}{h} \\ &= \lim_{h \to 0^-} \left( \frac{2a + b - a^2}{h} + 2 \right) = \begin{cases} 2 & \text{if } 2a + b - a^2 = 0 \\ \text{D.N.E.} & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0^+} \frac{2ah + h^2}{h} = \lim_{h \to 0^+} \big(2a + h\big) = 2a.$$

Thus these limits both exist and are the same if and only if $2a + b - a^2 = 0$ and $2 = 2a$. Solving these equatations, we see that $f$ is differentiable at $a$ if and only if $a = 1$ and $b = -1$. $\square$

## Assignment 7

Let $f$ be a differentiable function and let $a, b \in \mathbb{R}$ such that $a \neq b$.

(a) Suppose that $L_a(x) = L_b(x)$ for all $x \in \mathbb{R}$. Show that

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(a) + f'(b)}{2}. \tag{6}$$

(b) Conversely, suppose that (6) holds. Either prove that $L_a(x) = L_b(x)$ holds for all $x \in \mathbb{R}$, or provide a counterexample if it does not.

**Solution.**

(a) Recall that the linear approximation functions are defined as

$$L_a(x) = f(a) + f'(a)(x - a) \qquad \text{and} \qquad L_b(x) = f(b) + f'(b)(x - b)$$

for all $x \in \mathbb{R}$. Since $L_a(x) = L_b(x)$ holds for all $x \in \mathbb{R}$, it holds in particular when either $x = a$ or $x = b$. Thus, it holds that $L_a(a) = L_b(a)$ and $L_a(b) = L_b(b)$. Now,

$$\begin{aligned}
\frac{f'(a) + f'(b)}{2} &= \frac{f'(a)(b - a) - f'(b)(a - b)}{2(b - a)} \\
&= \frac{L_a(b) - f(a) - L_b(a) + f(b)}{2(b - a)} && \text{(since } L_a(b) = f(a) + f'(a)(b - a) \\
& && \quad \text{and } L_b(a) = f(b) + f'(b)(a - b)) \\
&= \frac{L_b(b) - f(a) - L_a(a) + f(b)}{2(b - a)} && \text{(since } L_a(a) = L_b(a) \text{ and } L_a(b) = L_b(b)) \\
&= \frac{f(b) - f(a) - f(a) + f(b)}{2(b - a)} && \text{(since } L_a(a) = f(a) \text{ and } L_b(b) = f(b)) \\
&= \frac{2f(b) - 2f(a)}{2(b - a)} \\
&= \frac{f(b) - f(a)}{b - a},
\end{aligned}$$

as desired.

(b) Consider the function defined by $f(x) = x^2$ and let $a = -1$ and $b = 1$. We have that $f'(x) = 2x$ for all $x$, and thus

$$\frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{1 - (-1)} = 0 \qquad \text{and} \qquad \frac{f'(a) + f'(b)}{2} = \frac{(-2) + 2}{2} = 0.$$

Hence (6) holds for this function and this choice of $a$ and $b$. However, the linear approximations are given by

$$L_a(x) = L_{-1}(x) = 1 - 2(x + 1) \qquad \text{and} \qquad L_b(x) = L_1(x) = 1 + 2(x - 1)$$

for all $x \in \mathbb{R}$. These are not the same function, since in particular we have that

$$L_a(1) = 1 - 2(1 + 1) = -3 \qquad \text{but} \qquad L_b(1) = 1 + 2(1 - 1) = 0$$

and thus $L_a(1) \neq L_b(1)$.

## Assignment 8

Let $c > 0$ be given and consider the curve defined by the equation $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Let $L$ be a tangent line to the curve with exactly one $x$-intercept and exactly one $y$-intercept. Prove that the sum of the $x$- and $y$-intercepts of $L$ is equal to $c$.

### Solution.

*Proof.* Let $(x_0, y_0)$ be the point on the curve (so that $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$) at which the line $L$ is tangent, and let $m$ be the slope of this line such that

$$L = \{(x, y) \in \mathbb{R}^2 \,:\, y = y_0 + m(x - x_0)\}.$$

(Note that we cannot have either $x_0 = 0$ or $y_0 = 0$, since otherwise the tangent line would have either infinitely many $y$-intercepts or infinitely many $x$-intercepts.)

We may use the method of implicit differentiation to find the slope of this line. On the curve, we have that

$$0 = \frac{d}{dx}\sqrt{c} = \frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

and thus

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

The slope of the tangent line at $(x_0, y_0)$ may be expressed as

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = -\left.\frac{\sqrt{y}}{\sqrt{x}}\right|_{(x,y)=(x_0,y_0)} = -\frac{\sqrt{y_0}}{\sqrt{x_0}}.$$

Let $x_1$ and $y_1$ denote the $x$- and $y$-intercepts of the line $L$, respectively. These are the values such that $(x_1, 0)$ and $(0, y_1)$ are on the line. That is, the values $x_1$ and $y_1$ must satisfy

$$y_1 = y_0 - mx_0 \qquad \text{and} \qquad 0 = y_0 + m(x_1 - x_0).$$

Rearranging, we find that

$$y_1 = y_0 - mx_0 = y_0 + \frac{\sqrt{y_0}}{\sqrt{x_0}}x_0 = y_0 + \sqrt{x_0 y_0}$$

and

$$x_1 = x_0 - \frac{1}{m}y_0 = x_0 + \frac{\sqrt{x_0}}{\sqrt{y_0}}y_0 = x_0 + \sqrt{x_0 y_0}.$$

Summing the $x$- and $y$-intercepts, we find that

$$x_1 + y_1 = x_0 + y_0 + 2\sqrt{x_0 y_0} = (\sqrt{x_0} + \sqrt{y_0})^2 = \sqrt{c}^2 = c,$$

as desired. □

## Assignment 9

Let $L \in \mathbb{R}$ with $L \neq 0$. Suppose that $f$ is a differentiable function such that $\lim\limits_{x\to\infty} f(x) = L$ and that $\lim\limits_{x\to\infty} f'(x)$ exists. Prove that $\lim\limits_{x\to\infty} f'(x) = 0$ using two different methods:

(a) using l'Hopital's rule.

(b) using the Mean Value Theorem.

**Solution.**

(a) If $L < 0$ then $\lim\limits_{x\to\infty} e^x f(x) = -\infty$. On the other hand, if $L > 0$ then $\lim\limits_{x\to\infty} e^x f(x) = +\infty$. In either case, we have that $\lim\limits_{x\to\infty} e^x f(x) = \pm\infty$ and that $\lim\limits_{x\to\infty} e^x = +\infty$, and thus $\lim\limits_{x\to\infty} \dfrac{e^x f(x)}{e^x}$ is an indefinite form of type $\pm\frac{\infty}{\infty}$. We may therefore apply l'Hopital's rule to find that

$$\begin{aligned}
L = \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{e^x f(x)}{e^x} \\
&= \lim_{x\to\infty} \frac{e^x(f(x) + f'(x))}{e^x} && \text{by l'Hopital's rule} \\
&= \lim_{x\to\infty} (f(x) + f'(x)) \\
&= \lim_{x\to\infty} f(x) + \lim_{x\to\infty} f'(x) \\
&= L + \lim_{x\to\infty} f'(x),
\end{aligned}$$

and thus $\lim\limits_{x\to\infty} f'(x) = 0$.

(b) We may construct an infinite sequence $\{c_n\}_{n\in\mathbb{N}}$ as follows. For every $n \in \mathbb{N}$, the Mean Value Theorem allows us to find a point $c_n \in (n, n+1)$ satisfying

$$f'(c_n) = \frac{f(n+1) - f(n)}{n+1-n} = f(n+1) - f(n).$$

Note that $n < c_n$ for every $n \in \mathbb{N}$ and thus $\lim_{n\to\infty} c_n = \infty$. By the Sequential Characterization of Limits and the fact that $\lim\limits_{x\to\infty} f'(x)$ exists, we have that

$$\begin{aligned}
\lim_{x\to\infty} f'(x) &= \lim_{n\to\infty} f'(c_n) \\
&= \lim_{n\to\infty} (f(n+1) - f(n)) \\
&= \lim_{n\to\infty} f(n+1) - \lim_{n\to\infty} f(n) \\
&= L - L \\
&= 0,
\end{aligned}$$

as desired.