Problem 1 (Aluffi problem VI.4.7).

Let $R$ be a commutative Noetherian ring, and let $M$ be a finitely generated module over $R$. Prove that $M$ admits a finite series

$$
(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M
$$
in which all quotients $M_{i+1}/M_i$ are of the form $R/p_i$ for some prime ideal $p_i$ of $R$.

Solution. We first prove a few lemmas (see Exercises VI.4.4 - VI.4.6 in Aluffi).

Lemma 1 (Exercise VI.4.4). Let $R$ be a commutative ring. Then $\text{Ann}_R(M)$ is an ideal of $R$ and $\text{Ann}_R(m)$ is an ideal of $R$ for each $m \in M$.

Proof. Let $a \in \text{Ann}_R(M)$, then $am = 0$ for all $m \in M$. For all $b \in R$ we have $(ab)m = (ba)m = b(am) = 0$ for all $m$, so $ba \in \text{Ann}_R(M)$. Similarly, if $a, a' \in \text{Ann}_R(M)$ then $(a + a')m = am + a'm = 0$ for all $m \in M$. Hence $\text{Ann}_R(M) \subseteq R$ is an ideal. Now let $m \in M$ and consider $\text{Ann}_R(m)$. Then $(ab)m = (ba)m = b(am) = 0$ for all $b \in R$ and $(a + a')m = am + a'm = 0$ for all $a, a' \in \text{Ann}_R(m)$. So $\text{Ann}_R(m) \subseteq R$ is an ideal.

Lemma 2 (Exercise VI.4.5). Let $R$ be a commutative ring and $M$ an $R$-module. Then for an ideal $I \subseteq R$ we have $I = \text{Ann}_R(m)$ for some $m \in M$ if and only if there is a submodule $N \subseteq M$ such that $N \cong R/I$.

Proof. Suppose that $I = \text{Ann}_R(m)$ for some $m \in M$ and define $N = \langle m \rangle$. Consider the $R$-module homomorphism

$$
\varphi: \langle m \rangle \longrightarrow R/I
$$

which is well-defined since if $a + I = a' + I$ then $(a - a') \in I$ so $(a - a')m = 0$ and thus $am = a'm$. This is injective, since $a + I = I$ if and only if $a \in I$ and thus $am = 0$. It is also surjective, since for all $a + I \in R/I$ we have $am \mapsto a + I$. So $\varphi$ is an isomorphism and thus $N \cong R/I$.

Now suppose that $M$ has a submodule $N \subseteq M$ such that $N \cong R/I$ for some $I$. Since $R/I$ is generated by $1 + I$, we have that $N = \langle m \rangle$ for some $m \in N$ such that $m \mapsto 1 + I$ under the isomorphism $N \cong R/I$. Then $I = \text{Ann}_R(m)$. Indeed, we have $am = 0$ if and only if $0 = am \mapsto a + 1 + I = I$ and thus $a \in I$.

Lemma 3 (Exercise VI.4.6). Let $R$ be a commutative ring and let $M$ be an $R$-module. Consider the family of ideals

$$
A_R(M) := \{ \text{Ann}_R(m) \mid m \in M, m \neq 0 \}.
$$

Then the maximal elements of $A_R(M)$ are prime ideals of $R$. 

Proof. Let $\mathfrak{m}$ be maximal in the family $\mathcal{A}_R(M)$. Then $\mathfrak{m} = \text{Ann}_R(m)$ for some $m \in M$ and if there is an ideal $I \in \mathcal{A}_R(M)$ such that $\mathfrak{m} \subseteq I$ then $\mathfrak{m} = I$. Suppose that $ab \in \mathfrak{m}$ for some elements $a, b \in R$ such that $a \notin \mathfrak{m}$. Then we have $abm = 0$ and assume without loss of generality that $am \neq 0$. Then $b(am) = 0$ implies that $b \in \text{Ann}_R(am)$. However, for all $r \in \text{Ann}_R(m) = \mathfrak{m}$ we have $rm = 0$ and thus $r(am) = a(rm) = 0$ so $r \in \text{Ann}_R(am)$. This tells us that $\mathfrak{m} \subseteq \text{Ann}_R(am)$, but $\mathfrak{m}$ is maximal and thus $\mathfrak{m} = \text{Ann}_R(am)$. Hence $b \in \mathfrak{m}$. So we have that $ab \in \mathfrak{m}$ if and only if $a \in \mathfrak{m}$ or $b \in \mathfrak{m}$ and thus $\mathfrak{m}$ is a prime ideal in $R$. \hfill \Box

Definition (Associated prime). Let $R$ be a commutative ring and $M$ be an $R$-module. An ideal $I \subset R$ is an associated prime of $M$ if $I = \text{Ann}_R(m)$ for some nonzero $m \in M$ and $I$ is a prime ideal of $R$. The set of associated primes of $M$ is denoted $\text{Ass}_R(M)$.

Corollary 4 (Exercise VI.4.6). If $R$ is a commutative Noetherian ring, then $\text{Ass}_R(M) \neq \emptyset$.

Proof. Suppose that there is no maximal element of $\mathcal{A}_R(M) := \{\text{Ann}_R(m) \mid m \in M, m \neq 0\}$. Then for every element in $\mathcal{A}$ we can find a larger element that contains it and there is an ascending chain of ideals of $R$

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

such that for each $i$ there is an element $m_i \in M$ with $I_i = \text{Ann}_R(m_i)$, a contradiction to the fact that $R$ is Noetherian. Hence there is at least one maximal element $\mathfrak{m} = \text{Ann}_R(m)$ of $\mathcal{A}_R(M)$. By the previous lemma, $\mathfrak{m}$ is a prime ideal of $R$ and thus $\mathfrak{m} \in \text{Ass}_R(M)$. \hfill \Box

We also note the definition of a Noetherian module and make a few useful notes about Noetherian modules.

Definition (Noetherian module). Let $R$ be a ring. An $R$-module $M$ is Noetherian if every submodule is finitely generated.

Proposition 5. Let $R$ be a ring and $M$ be a Noetherian $R$-module. Then $M$ satisfies the ascending chain condition. That is, if there is a sequence of submodules $M_i \subseteq M$ such that

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

then there is an $n \in \mathbb{N}$ such that $M_i = M_n$ for all $i \geq n$.

Proof. Suppose otherwise. Then without loss of generality we may assume that there is a sequence of submodules of $M$

$$M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots.$$ 

Then $N = \bigcup_{i=0}^{\infty} M_i$ is a submodule of $M$. Since $M$ is Noetherian, it is finitely generated so there is a finite set $a_1, \ldots, a_m$ of generators of $M$. But each $a_i$ must be contained in some $M_j$, so there is an $n$ such that $a_1, \ldots, a_m \in M_n$. But then $M_i = M$ for all $i \geq n$. \hfill \Box

Lemma 6. Let $M$ be an $R$-module, and let $N$ be a submodule of $M$. If $N$ and $M/N$ are both finitely generated, then $M$ is finitely generated.

Proof. Since $M/N$ is finitely generated there is a finite set of generators $a_1 + N, \ldots, a_n + N$ of $M/N$. Similarly there is a finite set of generators $b_1, \ldots, b_l$ of $N$. Let $m \in M$. Then

$$m + N = \sum_{i=1}^{n} r_i(a_i + N) = \left(\sum_{i=1}^{n} r_i a_i\right) + N.$$

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This implies that \( m - \sum_{i=1}^{n} r_i a_i \in N \) and thus
\[
m - \sum_{i=1}^{n} r_i a_i = s_1 b_1 + \cdots + s_l b_l
\]
for some \( s_j \in R \). Hence \( m = r_1 a_1 + \cdots + r_n a_n + s_1 b_1 + \cdots + s_l b_l \) so \( M \) is generated by \( a_1, \ldots, a_n, b_1, \ldots, b_l \)
and thus \( M \) is finitely generated.

**Proposition 7.** Let \( R \) be a ring, \( M \) an \( R \)-module and \( N \subset M \) a submodule. Then \( M \) is Noetherian if both \( N \) and \( M/N \) are Noetherian.

**Proof.** Let \( P \) be a submodule of \( M \), then we have to prove that \( P \) is finitely generated. Since \( P \cap N \) is a submodule of \( M \) and \( N \) is Noetherian, we have that \( P \cap N \subset N \) is finitely generated. By the Second Isomorphism Theorem for modules, we have that
\[
\frac{P}{P \cap N} \cong \frac{P + N}{N}
\]
and hence \( \frac{P}{P \cap N} \) is isomorphic to a submodule of \( M/N \). Since \( M/N \) is Noetherian, we have that \( \frac{P}{P \cap N} \) is finitely generated. Hence \( P \) is finitely generated since, by Lemma 6, \( P \cap N \) and \( \frac{P}{P \cap N} \) are both finitely generated. \( \square \)

**Proposition 8.** Let \( R \) be a Noetherian ring. Then an \( R \)-module \( M \) is Noetherian if and only if it is finitely generated.

**Proof.** If \( M \) is Noetherian then it is finitely generated since it is a submodule of itself and every submodule of \( M \) is finitely generated. So suppose that \( M \) is finitely generated, say by elements \( a_1, \ldots, a_n \in M \). Then there is a surjection \( R^{\oplus n} \twoheadrightarrow M \). By the previous proposition, it suffices to show that \( R^{\oplus n} \) is Noetherian as an \( R \)-module.

We prove this by induction. Note that \( R^{\oplus 1} = R \) is Noetherian by assumption. So suppose that \( R^{\oplus n} \) is Noetherian for some \( n \geq 1 \). Since \( R^{\oplus n} \) may be viewed as a submodule of \( R^{\oplus (n+1)} \) such that
\[
\frac{R^{\oplus (n+1)}}{R^{\oplus n}} \cong R,
\]
which is Noetherian, and \( R^{\oplus n} \) is Noetherian by assumption, it follows from Proposition 7 that \( R^{\oplus (n+1)} \) is Noetherian as well. \( \square \)

We can now prove the proposition of the problem.

**Proposition 9** (Problem statement). Let \( R \) be a commutative Noetherian ring, and let \( M \) be a finitely generated module over \( R \). Then \( M \) admits a finite series
\[
(0) = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M
\]
in which all quotients \( M_i/M_{i-1} \) are of the form \( R/p_i \) for some prime ideal \( p_i \) of \( R \).
Proof. First note that $\text{Ass}_R(M) \neq \emptyset$ by the Corollary 4 above, so there exists a prime ideal $p_1 \subset R$ such that $p_1 = \text{Ann}_R(m_1)$ for some $m_1 \in M$. By Lemma 2, there is a submodule $M_1 \subset M$ such that $M_1 \cong R/p_1$. Furthermore, we have $M_1/M_0 = M_1 \cong R/p_1$ where $M_0 = 0$.

We follow by induction. Suppose that for some $n \geq 1$ we have a series of submodules $M_0, M_1, \ldots, M_n$ of $M$ such that

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M$$

such that for each $i = 1, \ldots, n$ we have $M_i/M_{i-1} \cong R/p_i$ where each $p_i$ is prime. If $M/M_n \cong p$ for some prime ideal $p$ then we are done. Otherwise $M/M_n \neq 0$ and we have $\text{Ass}_R(M/M_n) \neq \emptyset$. So there is a submodule $M' \subset M/M_n$ such that $M' \cong R/p_{n+1}$ for some prime $p_{n+1}$. Set $M_{n+1}$ as the inverse image of $M'$ such that $M_{n+1}/M_n = M' \cong R/p_{n+1}$.

Since $R$ is Noetherian and $M$ is finitely generated, $M$ is Noetherian as an $R$-module. Thus by Proposition 5 we must have that the sequence

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$$

eventually terminates with $M_m = M$ for some $m \in \mathbb{N}$. This proves the claim. \qed
**Problem 2** (Aluffi problem VI.4.13).

Let $R$ be a commutative ring. A tuple $(a_1, \ldots, a_n)$ of elements in $R$ is a regular sequence if $a_1$ is a non-zero-divisor in $R$ and each $a_i$ is a non-zero-divisor modulo $(a_1, \ldots, a_{i-1})$ for $i > 1$.

For $a, b \in R$, consider the following complex of $R$-modules:

$$
0 \longrightarrow R \xrightarrow{d_2} R \oplus R \xrightarrow{d_1} R \xrightarrow{\pi} R_{(a,b)} \longrightarrow 0
$$

where $\pi$ is the canonical projection, $d_1(r,s) = ra + sb$ and $d_2(t) = (bt, -at)$. That is, $d_1$ and $d_2$ correspond to the matrices

$$(a \ b) \quad \text{and} \quad \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}.$$  

i) Prove that this is indeed a complex for every $a$ and $b$.

ii) Prove that if $(a,b)$ is a regular sequence, then this complex is exact.

The complex $(\ast)$ is called the Kozul complex of $(a,b)$. Thus, when $(a,b)$ is a regular sequence, the Kozul complex provides us with a free resolution of the module $R/(a,b)$.

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**Solution.** Proof. .

i) Note that $d_1(r,s) = ar + sb \in (a,b)$ for all $r, s \in R$, hence $\pi(d_1(r,s)) = 0$ and thus $\text{im}(d_1) \subset \text{ker}\pi$. Similarly, 

$$d_1(d_2(t)) = d_1(bt, -at) = bta - atb = 0$$

for all $t$ and thus $\text{im}(d_2) \subset \text{ker}(d_1)$. So $(\ast)$ is indeed a complex.

ii) We examine the complex at each spot to determine exactness. Since the sequence $(a,b)$ is regular, we have that $a$ is a non-zero-divisor and $b$ is a non-zero-divisor modulo $(a)$. That is, $bc \notin (a)$ for all $c \notin (a)$.

- The complex is clearly exact at $R \xrightarrow{\pi} R_{(a,b)} \longrightarrow 0$ by surjectivity of $\pi$.
- Let $t \in \text{ker}\pi = (a,b)$, then $t = ra + sb$ for some $r, s \in R$. Hence $t = d_1(r,s)$ and thus $\text{im}(d_1) = \text{ker}\pi$ so the complex is exact in the second spot.
- Let $(r,s) \in \text{ker}(d_1)$. Then $d_1(r,s) = ra + sb = 0$ such that $sb = -ra$ and thus $sb \in (a)$. But $b$ is a non-zero-divisor modulo $(a)$. Hence $sb \in (a)$ implies $s \in (a)$ and thus $s = t'a$ for some $t' \in R$. So we have

$$ra + (t'a)b = 0 \implies a(r + t'b) = 0.$$  

But $a$ is a non-zero-divisor and thus $(r + t'b) = 0$, so $r = -t'b$. Setting $t = -t' \in R$ yields $(r,s) = (bt, -at)$ and thus $(r,s) \in \text{im}(d_2)$. Hence $\text{im}(d_2) = \text{ker}(d_2)$ so we have that the complex is exact at $R \xrightarrow{d_2} R \oplus R \xrightarrow{d_1} R$.

- Finally, the complex is exact in the last spot since $d_2$ is injective. Indeed, for some $t \in R$ suppose that $d_2(t) = (bt, -at) = (0,0)$. Then $at = 0$, but $a$ is a non-zero-divisor and thus $t = 0$.  

\qed
Problem 3 (Aluffi problem VI.5.5).

Recall that a commutative ring is local if it has a single maximal ideal \( m \). Let \( R \) be a local ring and let \( M \) be the direct summand of a finitely generated free \( R \)-module. That is, there exists an \( R \)-module \( N \) such that \( M \oplus N \) is a free \( R \)-module.

i) Choose elements \( m_1, \ldots, m_r \in M \) whose cosets mod \( mM \) are a basis of \( M/mM \) as a vector space over the field \( R/m \). By Nakayama’s lemma, \( M = \langle m_1, \ldots, m_r \rangle \).

ii) Obtain a surjective homomorphism \( \pi: R^{\oplus r} \longrightarrow M \).

iii) Show that \( \pi \) splits, giving an isomorphism \( R^{\oplus r} \cong M \oplus \ker \pi \).

iv) Show that \( \ker \pi/m \ker \pi = 0 \). Use Nakayama’s lemma to deduce that \( \ker \pi = 0 \).

v) Conclude that \( M \cong R^{\oplus r} \) and this \( M \) is in fact free.

Summarizing, over a local ring, every direct summand of a finitely generated free \( R \)-module is free. (Contrast this fact with Proposition VI.5.1, which shows that every submodule of a finitely generated free module is free.)

Solution. Recall the two statements of Nakayama’s Lemma that we will use:

Lemma 10 (Nakayama). Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Suppose \( I \subset J \) is an ideal of \( R \) that is contained in the Jacobson radical \( J \) of \( R \).

1. If \( m_1, \ldots, m_r \) have images in \( M/IM \) that generate it as an \( R/I \)-module, then \( m_1, \ldots, m_r \) generate \( M \) as an \( R \)-module.

2. If \( M/IM = 0 \) then \( M = 0 \).

Since the Jacobson radical is the intersection of all maximal ideals of \( R \), in this case we have that \( J = m \). We now prove the problem statement.

i) First note that \( M \) is finitely generated. Indeed, since there is an \( R \)-module \( N \) such that \( R^{\oplus n} \cong M \oplus N \), there is a surjective map \( R^{\oplus n} \longrightarrow M \) given by the projection map

\[
R^{\oplus n} \cong M \oplus N \xrightarrow{\pi_M} M.
\]

Since \( m \subset R \) is maximal, \( R/m \) is a field. Note that \( M/mM \) has an \( R/m \)-module structure given by

\[(r + m)(m + mM) = rm + mM\]

and thus \( M/mM \) is an \( R/m \)-vector space. Also note that \( M/mM \) is finitely generated since \( M \) is. So \( M/mM \cong (R/m)^{\oplus r} \) for some integer \( r \) and thus has a basis given by \( m_1 + mM, \ldots, m_r + mM \). By Nakayama’s lemma, \( m_1, \ldots, m_r \) also generates \( M \).

ii) Since \( m_1, \ldots, m_r \) generate \( M \), we have a surjective homomorphism

\[
\pi: R^{\oplus r} \longrightarrow M
\]

\[(a_1, \ldots, a_r) \mapsto a_1m_1 + \cdots + a_rm_r.\]
iii) Since \( \pi \) is surjective and \( M \oplus N = R^{\oplus n} \) is free (and thus projective), there is an \( R \)-module homomorphism \( R^{\oplus n} \xrightarrow{\beta} R^{\oplus r} \) such that the diagram

\[
\begin{array}{ccc}
R^{\oplus r} & \xrightarrow{\pi} & M \\
\downarrow{\beta} & & \downarrow{\pi_M} \\
R^{\oplus n} & \cong & M \oplus N
\end{array}
\]

commutes (see Exercise III.6.9 in Aluffi), where \( M \oplus N \xrightarrow{\pi_M} M \) is the projection that maps the direct sum onto \( M \). Note that \( \pi_M \) has the natural right-inverse given by \( M \xrightarrow{\iota_M} M \oplus N \) such that the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker \pi \\
\beta & \swarrow{\iota_M} & \downarrow{\pi_M} \\
R^{\oplus n} & \cong & M \oplus N
\end{array}
\]

commutes. Hence \( \pi \) has a right-inverse with \( \pi \circ \beta \circ \iota_M = \text{id}_M \) and thus the exact sequence

\[
0 \longrightarrow \ker \pi \longrightarrow R^{\oplus r} \longrightarrow M \longrightarrow 0 \quad (**)
\]

splits. This is equivalent to saying that \( R^{\oplus r} \cong M \oplus \ker \pi \).

iv) Tensoring \( - \otimes_R R/m \) with \( R^{\oplus r} \) and \( M \oplus \ker \pi \) yields the \( R \)-modules

\[
R^{\oplus r} \otimes_R R/m \cong (R/m)^{\oplus r} \quad \text{and} \quad (M \oplus \ker \pi) \otimes_R R/m \cong (M/mM) \oplus (\ker \pi/m \ker \pi),
\]

which all finite-dimensional as \( R/m \)-vector spaces. We have the exact sequence

\[
0 \longrightarrow \ker \pi/m \ker \pi \longrightarrow (R/m)^{\oplus r} \xrightarrow{\pi} M/mM \longrightarrow 0.
\]

But \( M/mM \cong (R/m)^{\oplus r} \) as an \( (R/m) \)-vector space, so we have that \( \ker \pi/m \ker \pi = 0 \). Thus \( \ker \pi = 0 \) by Nakayama’s lemma.

v) Since \( \ker \pi = 0 \), we have the exact sequence

\[
0 \longrightarrow R^{\oplus r} \xrightarrow{\pi} M \longrightarrow 0
\]

and thus \( M \cong R^{\oplus r} \) so \( M \) is free as desired.