# Riddler March 12, 2021: The Biggest $\pi$

Mark Girard

March 15, 2021

Consider the following puzzle about baking a pie in honour of Pi Day.

**Question 1.** *You have a sheet of crust laid out in front of you. After baking, your pie crust will be a cylinder of uniform thickness (or rather, thinness) with delicious filling inside. To maximize the volume of your pie, what fraction of your crust should you use to make the circular base (i.e., the bottom) of the pie?*

In solving this puzzle, I assumed that the thickness of the crust needed to be taken into consideration when finding the optimal pie geometry. One can recover the answer in the case of an infinitesimally thin crust by taking the limit of my solution as the thickness tends to zero.

Suppose we are given some volume of dough that we are to use for the crust, which will have a uniform thickness when we are finished. Let's begin by defining a few variables.

- $v_{\text{crust}}$ is the volume of the dough to be used for the crust.
- $t$ is the uniform thickness of the crust.
- $r$ is the radius of the constructed cylindrical pie.
- $h$ is the height of the pie.
- $v_{\text{fill}}$ and $v_{\text{pie}}$ are the volumes of the filling and total pie, respectively, such that

$$\pi r^2 h = v_{\text{pie}} = v_{\text{crust}} + v_{\text{fill}}. \tag{1}$$

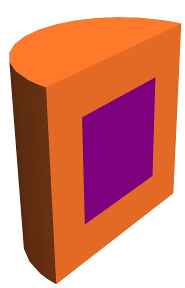


Figure 1: Example geometry of a pie (sliced through the middle). The complete pie consists of a cylinder of filling (purple) inside a cylindrical shell of crust having uniform thickness (orange).

The geometry we are aiming for is something like the nested cylinders shown in Figure 1. The interior (i.e., the filling) is a cylinder having height $h - 2t$ and radius $r - t$, and thus

$$v_{\text{fill}} = \pi(r-t)^2(h-2t).$$

Clearly we must have both $r \geq t$ and $h \geq 2t$ for the pie to be geometrically feasible. We may express the volume of the crust as

$$\begin{aligned} v_{\text{crust}} &= v_{\text{pie}} - v_{\text{fill}} \\ &= \pi r^2 h - \pi(r-t)^2(h-2t). \end{aligned} \tag{2}$$

The problem now is to maximize $v_{\text{fill}}$ subject to constant $v_{\text{crust}}$ and thickness $t$, by finding the optimal vales $r$ and $h$. That is:

$$\begin{aligned} \text{maximize:} \quad & \pi r^2 h \\ \text{subject to:} \quad & r \geq t,\ h \geq 2t \\ & \pi r^2 h - \pi(r-t)^2(h-2t) = v_{\text{crust}}, \end{aligned} \tag{3}$$

where $v_{\text{crust}}$ and $t$ are some fixed constants. The constraint in (3) can be simplified to

$$h(2r-t) + 2(r-t)^2 = \frac{v_{\text{crust}}}{\pi t}.$$

By the method of Lagrange multipliers, the optimal value will occur for values $r$ and $h$ when there is some constant $\lambda$ satisfying

$$\begin{pmatrix} 2\lambda r h \\ \lambda r^2 \end{pmatrix} = \begin{pmatrix} 2h + 4(r-t) \\ 2r - t \end{pmatrix}.$$

Equating the second components, we find that $\lambda r = (2r-t)/r$. Substituting this into the equation of the first components yields that the optimal value will occur when

$$h(2r-t) = hr + 2r(r-t),$$

which simplifies to

$$(r-t)(2r-h) = 0.$$

Since the volume of the filling is zero when $r = t$, we find that the maximal value must occur when $h = 2r$. That is, the geometry of the optimal pie will always have height exactly equal to twice the radius!

It remains now to find an expression for the fraction of dough used for the crust in an optimally shaped pie. Substituting $h = 2r$ into the expression for $v_{\text{crust}}$ in (2) yields

$$\begin{aligned} v_{\text{crust}} &= 2\pi r^3 - 2\pi(r-t)^3 \\ &= 2\pi(3r^2 t - 3rt^2 + t^3). \end{aligned}$$

Meanwhile, the volume of crust used for the base is

$$v_{\text{base}} = \pi r^2 t.$$

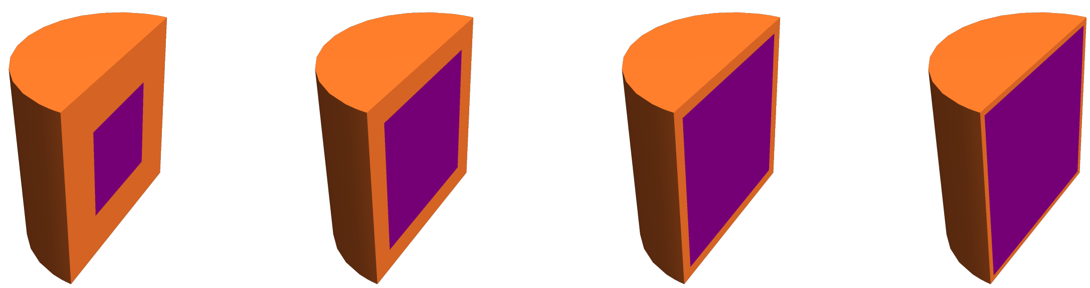
Thus the fraction of crust used for the base is

$$x = \frac{v_{\text{base}}}{v_{\text{crust}}} = \frac{r^2}{2(3r^2 - 3rt + t^2)} = \frac{1}{6 - 6s + 2s^2}$$

where we define $s = t/r$. In the limit of a very thin crust (i.e., $t \to 0$), we find that the optimal ratio of crust to use for the base is $1/6$.

## Visualizing optimal pies

Finally, lets use some graphing software to visualize some of these pies. Figure 2 displays the optimal pie shape (sliced through the middle to display the filling) for pies having fixed radius $r = 1$ and varying choices for the thickness $t \in [0, 1]$. As stated in the previous section, the optimal pie always has height equal to twice its radius.



(a) $t = \frac{1}{2}, x = \frac{2}{7}$. (b) $t = \frac{1}{4}, x = \frac{8}{37}$. (c) $t = \frac{1}{8}, x = \frac{32}{169}$. (d) $t = \frac{1}{16}, x = \frac{128}{721}$.

Figure 2: Visualizing optimal pies (sliced vertically through the middle) for different values of crust thickness $t$ given a fixed radius $r = 1$. The optimal pie always has height equal to twice the radius. The crust is orange and the filling is purple. The fraction of crust comprising the base is $x = 1/(6 - 6t + 2t^2)$. In the limit as $t \to 0$, the optimal fraction of crust to use for the base tends to $x \to \frac{1}{6}$.