Riddler March 19, 2021: Can You Find An Extra Perfect Square?

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March 21, 2021

Question 1. *For some perfect squares, when you remove the last digit, you get another perfect square. For example, when you remove the last digit from 256 (162), you get 25 (52).*

The first few squares for which this happens are 16, 49, 169, 256 and 361. What are the next three squares for which you can remove the last digit and get a different perfect square? How many more can you find? (Bonus points for not looking this up online or writing code to solve it for you! There are interesting ways to do this by hand, I swear.)

Extra credit: Did you look up the sequence and spoil the puzzle for yourself? Good news, there's more! In the list above, 169 (132) is a little different from the other numbers. Not only when you remove the last digit do you get a perfect square, 16 (42), but when you remove the last two digits, you again get a perfect square: 1 (12). Can you find another square with both of these properties?

The goal is to find integers $x, y \in \mathbb{N}$ such that $x^2 = 10y^2 + r$ for some $r \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We must have that $x^2 \equiv r \mod 10$, but the only possible quadratic residues modulo 10 are $r \in \{0, 1, 4, 5, 6, 9\}$. Note that $r \neq 0$ unless $x = y = 0$, as 10 is not square (as we cannot otherwise have $x^2 = 10y^2$). It is also not possible to have $r = 5$ (as this would require $5|x$ and $2 \not| x$ such that $x^2 \equiv 25 \mod 100$, but this would imply $y^2 \equiv 2 \mod 10$, which is impossible). Therefore, we are looking for solutions to

$$x^2 - 10y^2 = r \qquad \text{where} \quad r \in \{1, 4, 6, 9\}.$$

From here, things get a bit more difficult. We can make use of some well known analysis of solutions to the generalized Pell equation. It turns out, however, that answering the extra credit problem is much easier!

Extra credit solution

For the extra credit problem, we are looking for triples of integers $x, y, z \in \mathbb{N}$ that satisfy

$$x^2 = 10y^2 + r \qquad \text{and} \qquad y^2 = 10z^2 + s$$

for some residues $r, s \in \{1, 4, 6, 9\}$. A solution here must therefore satisfy

$$x^2 = (10z)^2 + 10r + s$$

for some $r, s \in \{1, 4, 6, 9\}$. Let x and y comprise such a solution and let $k = x - 10z$ so that $k \geq 1$. It follows that $20kz + k^2 = 10r + s$ and thus $20z < 99$ (since we must have $10r + s \leq 99$), which implies $z < 5$. To find all solutions, it therefore suffices to check the values $z \in \{1,2,3,4\}$ and $r,s \in \{1,4,6,9\}$ for which the numbers $100z^2 + 10r + s$ and $10z^2 + r$ are perfect squares. Here is some python code that implements this search.

```
import math

def is_square(n):
    return math.isqrt(n)**2 == n

for z in [1,2,3,4]:
    for r in [1,4,6,9]:
        for s in [1,4,6,9]:
            if is_square(100*z**2 + 10*r + s) and is_square(10*z**2 + r ):
                print(100*z**2 + 10*r + s)
```

We see that the only solution is 169.

Pell's equation and $\mathbb{Z}[\sqrt{10}]$

We want to look for solutions to the following equations:

$$x^2 - 10y^2 = 1$$
$$x^2 - 10y^2 = 4$$
$$x^2 - 10y^2 = 6$$
$$x^2 - 10y^2 = 9$$

These are examples of a *generalized Pell equation*, which have been extensively studied. Solutions can be found by studying the ring $\mathbb{Z}[\sqrt{10}] = \{x + y\sqrt{10} : x, y \in \mathbb{Z}\}$. This ring is a *Euclidean domain* with Euclidean norm function $N : \mathbb{Z}[\sqrt{10}] \to \mathbb{Z}$ defined by

$$N(x + y\sqrt{10}) = (x + y\sqrt{10})(x - y\sqrt{10}) = x^2 - 10y^2$$

for all $x, y \in \mathbb{Z}$ which satisfies the following conditions:

- $N(u) = 0$ if and only if $u = 0$
- $N(uv) = N(u)N(v)$ for all $u, v \in \mathbb{Z}[\sqrt{10}]$

An element $u \in \mathbb{Z}[\sqrt{10}]$ is a *unit* if $N(u) = 1$. There exists a *fundamental unit* $u \in \mathbb{Z}[\sqrt{10}]$ such that every other unit $v \in \mathbb{Z}[\sqrt{10}]$ is of the form $v = u^k$ for some $k \in \mathbb{N}$. It can be worked out that this unit is: $u = 19 + 6\sqrt{10}$.

Similarly, for $n \in \mathbb{N}$ with $n > 1$, all solutions to

$$x^2 - 10y^2 = n$$

are of the form vu^k for some $k \in \mathbb{N}$, where v is one of finitely many *fundamental solutions* to this equation.

Solutions to $x^2 - 10y^2 = 1$

The fundamental unit is $u = 19 + 6\sqrt{10}$ and all solutions are of the form $x^2 - 10y^2 = u^k$ for some $k \in \mathbb{N}$. The first few solutions are:

k	$(19+6\sqrt{10})^k = x + y\sqrt{10}$	x^2
1	$19 + 6\sqrt{10}$	361
2	$721 + 228\sqrt{10}$	519841
3	$27379 + 8658\sqrt{10}$	749609641
4	$1039681 + 328776\sqrt{10}$	1080936581761

Solutions to $x^2 - 10y^2 = 4$

There is only one fundamental solution: 2. All other solutions are of the form $2(19 + 6\sqrt{10})^k$ for some $k \in \mathbb{N}$. The first few solutions are:

k	$2(19+6\sqrt{10})^k = x + y\sqrt{10}$	x^2
1	$38 + 12\sqrt{10}$	1444
2	$1442 + 456\sqrt{10}$	2079364
3	$54758 + 17316\sqrt{10}$	2998438564
4	$2079362 + 657552\sqrt{10}$	4323746327044

Solutions to $x^2 - 10y^2 = 6$

There are two fundamental solutions: $4 \pm \sqrt{10}$. Note that $4 - \sqrt{10}$ is not a true solution, since we require $x, y \geq 0$. But multiplying this by the fundamental unit yields the solution

$$(4 - \sqrt{10})(19 + 6\sqrt{10}) = 16 + 5\sqrt{10}.$$

All other solutions are of the form

$$(4 + \sqrt{10})(19 + 6\sqrt{10})^k \qquad \text{or} \qquad (16 + 5\sqrt{10})(19 + 6\sqrt{10})^k$$

for some $k \in \mathbb{N}$. The first few solutions are:

k	$(4+\sqrt{10}(19+6\sqrt{10})^k = x + y\sqrt{10}$	x^2
0	$4 + \sqrt{10}$	16
1	$136 + 43\sqrt{10}$	18496
2	$5164 + 1633\sqrt{10}$	26666896
3	$196096 + 62011\sqrt{10}$	38453641216

k	$(4-\sqrt{10}(19+6\sqrt{10})^k = x + y\sqrt{10}$	x^2
1	$16 + 5\sqrt{10}$	256
2	$604 + 191\sqrt{10}$	364816
3	$22936 + 7253\sqrt{10}$	526060096
4	$870964 + 275423\sqrt{10}$	758578289296

Solutions to $x^2 - 10y^2 = 9$

There are three fundamental solutions: 3 and $7 \pm 2\sqrt{10}$. As before $7 - 2\sqrt{10}$ is not a true solution, but multiplying this by the fundamental unit yields the solution

$$(7 - 2\sqrt{10})(19 + 6\sqrt{10}) = 13 + 4\sqrt{10}.$$

All other solutions are of the form

$$3(19 + 6\sqrt{10})^k, \qquad (7 + 2\sqrt{10})(19 + 6\sqrt{10})^k \qquad \text{or} \qquad (13 + 4\sqrt{10})(19 + 6\sqrt{10})^k$$

for some $k \in \mathbb{N}$. The first few solutions are:

k	$3(19 + 6\sqrt{10})^k = x + y\sqrt{10}$	x^2
1	$57 + 18\sqrt{10}$	3249
2	$2163 + 684\sqrt{10}$	4678569
3	$82137 + 25974\sqrt{10}$	6746486769
4	$3119043 + 62011\sqrt{10}$	9728429235849

k	$(7 + 2\sqrt{10}(19 + 6\sqrt{10})^k = x + y\sqrt{10}$	x^2
0	$7 + 2\sqrt{10}$	49
1	$253 + 80\sqrt{10}$	64009
2	$9607 + 3038\sqrt{10}$	92294449
3	$364813 + 115364\sqrt{10}$	133088524969

k	$(13 + 4\sqrt{10}(19 + 6\sqrt{10})^k = x + y\sqrt{10}$	x^2
0	$13 + 4\sqrt{10}$	169
1	$487 + 154\sqrt{10}$	237169
2	$18493 + 5848\sqrt{10}$	341991049
3	$702247 + 222070\sqrt{10}$	493150849009