Riddler March 36, 2021:
Free throws

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Question 1. *The rules for men's basketball in the Riddler Collegiate Athletic Association's (RCAA) are a little different from those in the NCAA. In the RCAA, when taking free throws, a player must earn each additional foul shot by making the previous one. In other words, a player can take a second shot if they make the first, and they can take a third shot if they make the second.*

Suppose a player on your team has a known shooting profile: Their probability of making the first free throw is p, their probability of making the second is q, and their probability of making the third is r, such that no two of these probabilities are equal. Meanwhile, their expected number of points made for any given three-point foul (which can be computed from p, q and r) is also known.

What is the greatest number of distinct shooting profiles that are made up of these three different probabilities—p, q and r, in some order for the three shots—that can result in the same overall expected number of points?

Supposing the probabilities of making the first, second, and third shots are p, q, and r respectively, there are four possible outcomes:

- Miss the first shot with probability $1 - p$.
- Make the first shot but miss the second with probability $p(1 - q)$.
- Make the first two shots but miss the third with probability $pq(1 - r)$.
- Make all three shots with probability pqr.

The expected value for the number of shots made is therefore:

$$\mathrm{E} = 0 \cdot (1 - p) + 1 \cdot p(1 - q) + 2 \cdot pq(1 - r) + 3 \cdot pqr$$

which simplifies nicely to

$$\mathrm{E} = p + pq + pqr.$$

For every tuple (p', q', r') that is a permutation of (p, q, r), it holds that $p'q'r' = pqr$ and thus the permutation of the probabilities yields the same expected number of shots made in a three-shot free-throw attempt if and only if $p'(1 + q') = p(1 + q)$. The problem can therefore be rephrased as follows.

Problem 2. Maximize the number of distinct permutations (p', q', r') for some choice of tuple (p, q, r) of distinct probabilities $p, q, r \in [0, 1]$ that satisfy $p'(1 + q') = p(1 + q)$.

It is clear that the optimal number is not less than 2. Indeed, we can take a rather trivial solution having $p = 0$ and any choice of distinct values $q, r \in (0, 1]$. (Namely, if the player never makes the first shot, it does not matter what probabilities they have of making the second and third shots. The expected number of shots is always zero!) Less trivial solutions can also be easily constructed. For example, consider the probabilities $(p, q, r) = (1/2, 1/3, 1)$ and permutation $(p', q', r') = (q, r, p)$ which have expected numbers of shots given by

$$\begin{aligned} p + pq + pqr &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \cdot 1 &&= \frac{5}{6} \\ \text{and} \quad p' + p'q' + p'q'r' &= \frac{1}{3} + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} &&= \frac{5}{6}. \end{aligned}$$

However, no further permutation of these probabilities yields the same expected value.

We now show that it is impossible to have a feasible solution with three distinct permutations. Let $p, q, r \in [0, 1]$ be distinct, and suppose (p', q', r') and (p'', q'', r'') are distinct non-trivial permutations of (p, q, r) satisfying

$$p(1 + q) = p'(1 + q') = p''(1 + q''). \tag{1}$$

If $p = p'$ then either $p = 0$, in which case it must also hold that $p'' = 0$, or $q = q'$ and thus $r = r'$. Both cases lead to a contradiction, as the three triples must be distinct, and thus it must be the case that $p \neq p'$. Analogously, it must also be the case that $p' \neq p''$ and $p \neq p''$. Similarly, if $q = q'$ then it must be the case that $p = p'$ and $r = r'$, which is a contradiction. We may analogously conclude that $q \neq q'$, $q \neq q''$, and $q' \neq q''$. It follows that we must also have $r \neq r'$, $r \neq r''$, and $r' \neq r''$. This leaves only two possibles choices for (p', q', r') and (p'', q'', r''). (That is, both must be derangements of (p, q, r) and of each other.)

We may therefore suppose without loss of generality that these permutations are $(p', q', r') = (q, r, p)$ and $(p'', q'', r'') = (r, p, q)$. From (1), we have

$$p(1 + q) = q(1 + r) = r(1 + p),$$

which is equivalent to

$$p - r = p(r - q), \qquad q - p = q(p - r), \qquad \text{and} \qquad r - q = r(q - p).$$

It follows that

$$p - r = p(r - q) = pr(q - p) = pqr(p - r)$$

and thus $(p - q)(1 - pqr) = 0$, which implies that either $p = q$ or $p = q = r = 1$, which is a contradiction to the assumption that $p, q, r \in [0, 1]$ are all distinct.