Riddler April 2, 2021:
Riddle of the Sphinx

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Question 1. *You will be asked four seemingly arbitrary true-or-false questions by the Sphinx on a topic about which you know absolutely nothing. Before the first question is asked, you have exactly \$1. For each question, you can bet any non-negative amount of money that you will answer correctly. That is, you can bet any real number (including fractions of pennies) between zero and the current amount of money you have. After each of your answers, the Sphinx reveals the correct answer. If you are right, you gain the amount of money you bet; if you are wrong, you lose the money you bet.*

However, there's a catch. (Isn't there always, with the Sphinx?) The answer will never be the same for three questions in a row.

With this information in hand, what is the minimum amount of money you can be sure that you'll win, no matter what the answers wind up being?

Extra credit: This riddle can be generalized so that the Sphinx asks N questions, such that the answer is never the same for Q questions in a row. What are your maximum guaranteed winnings in terms of N and Q?

Let's go straight to solving the extra credit version. When playing this game against the sphinx, the maximum amount of guaranteed winnings at every step of the way depends only on the current state of the game. The only pieces of information that matter for determining this are:

- How many questions are remaining in the game?
- What is the length of the most recent streak of identical answers?

Just before the very first question is asked—when no questions have been answered yet—the length of the most recent streak is zero. In all other states of the game after the first question has been answered, the length of the most recent streak will be at least one but never more than $Q-1$. Let $r_{n,k}$ denote the maximum guaranteed return on investment (on the current amount of money held) that can be achieved, given that there are n questions remaining in the game and that the length of the most recent streak of identical answers is k. A few observations:

- We can only have $k=0$ before the first question and we can never have $k \geq Q$.
- When there are no questions left, there is no chance for more gains and thus we must have $r_{0,k}=0$ for all $k \in \{1,\ldots,Q-1\}$.

- When $k = Q - 1$ and there is at least one question remaining (i.e., $n \geq 1$), we are guaranteed to double our money (by betting it all on the answer being different from the previous answer)! The minimum guaranteed return on investment on the current amount of funds held is therefore $r_{n,Q-1} = 1$.

The maximal guaranteed return on investment for other states of the game can be determined inductively as follows. Suppose there are n questions remaining and the length of the most recent streak is k, where $n \geq 1$ and $k \in \{1, \ldots, Q-1\}$. Let $w \in [-1, 1]$ denote the bet placed on what the next answer will be, where $|w|$ is the fraction of the amount of money currently held that is wagered and the sign of w determines whether we bet on the next answer being different from or the same as the previous answer. (We let $w > 0$ denote a bet that the next answer will be different and $w < 0$ denote a bet that the next answer will be the same). Suppose we currently hold x dolars. After the next answer is revealed, the number of remaning questions is now $n - 1$, but there are two outcomes:

- The next answer is different from the previous one, in which case the length of the current streak is now 1 and we have $x(1 + w)$ dollars. With $n - 1$ questions remaning and a current streak length of 1, the largest amount of money we are guaranteed to have by the end of the game is now equal to $x(1 + w)(1 + r_{n-1,1})$.
- The next answer is the same as the previous one, in which case the length of the current streak is now $k + 1$ and we have $x(1 - w)$ dollars. With $n - 1$ questions remaning and a current streak length of $k + 1$, the largest amount of money we are guaranteed to have by the end of the game is now equal to $x(1 - w)(1 + r_{n-1,k+1})$.

The largest *guaranteed* return on investment the end of the game is the smaller of these two numbers:

$$(1 + w)(1 + r_{n-1,1}) - 1 \qquad \text{and} \qquad (1 - w)(1 + r_{n-1,k+1}) - 1.$$

The optimal wager is the value w that maximizes the minimum of these two numbers. The largest guaranteed return on investment at the state of the game with n questions remaining and a current streak of k (with $k < Q - 1$) therefore satisfies

$$1 + r_{n,k} = \max_{w \in [-1,1]} \left\{ \min\left\{ (1 + w)(1 + r_{n-1,1}), (1 - w)(1 + r_{n-1,k+1}) \right\} \right\}.$$

Note that the largest guaranteed winnings rate must be at least zero (i.e., $r_{n,k} \geq 0$) as we could always choose to always bid nothing.

Consider now the optimization problem

$$\begin{aligned} \text{maximize :} \qquad & \min\{(1 + w)a, (1 - w)b\} \\ \text{subject to :} \qquad & w \in [-1, 1] \end{aligned}$$

for fixed values $a, b \geq 0$. The optimal value will occur either at the boundary (i.e., $w = \pm 1$) or when the two quantities are equal (i.e., $(1 + w)a = (1 - w)b$). Choosing $w = \pm 1$ yields an answer of zero, so we conclude that the optimal value occurs when $(1 + w)a = (1 - w)b$, which occurs when

$$w = \frac{b - a}{b + a} = \frac{2b}{a + b} - 1,$$

and thus the optimial value is $\frac{2ab}{a+b}$, which we may express as $2\left(\frac{1}{a}+\frac{1}{b}\right)^{-1}$.

Therefore, the sequence recursively satisfies

$$\frac{1}{1+r_{n,k}} = \frac{1}{2}\left(\frac{1}{1+r_{n-1,1}} + \frac{1}{1+r_{n-1,k+1}}\right).$$

Finally, at the initial state of the game when $n = N$ and $k = 0$, regardless of the outcome of the first question the following state of the game will have $n-1$ questions remaining and a current streak of 1. The largest guaranteed return therefore satisfies

$$1 + r_{N,0} = \max_{w\in[-1,1]}\{\min\{(1+w)(1+r_{N-1,1}),(1-w)(1-r_{N-1,1})\}\},$$

and the optimal value is clearly achieved when $w = 0$. Hence $r_{N,0} = r_{N-1,1}$.

To recap, finding the solution to the riddle amounts to solving the recurrence relation determined by the following rules:

- $r_{0,k} = 0$ for all $k \in \{1,\ldots,Q-1\}$.
- $r_{n,Q-1} = 1$ for all $n \geq 1$.
- For $n \geq 1$ and $k \in \{1,\ldots,Q-2\}$, we have $\frac{1}{1+r_{n,k}} = \frac{1}{2}\left(\frac{1}{1+r_{n-1,1}} + \frac{1}{1+r_{n-1,k+1}}\right)$.

The largest guaranteed return on investment at the end of a game with N questions is $r_{N-1,1}$.

For simplicity, we may introduce a change of variables and define the sequence

$$a_{n,k} = \frac{2^n}{1+r_{n,k}}$$

for $n \geq 0$, which satisfies

$$a_{n,k} = a_{n-1,1} + a_{n-1,k+1}$$

whenever $k \in 1,\ldots,Q-2$ and satisfies $a_{n,Q-2} = 2^{n-1}$ when $n \geq 1$. Moreover $a_{0,k} = 1$ for all k. If $Q \geq 3$, then

$$a_{1,1} = a_{0,1} + a_{0,1} = 1 + 1 = 2.$$

In general, if $n < Q$, then

$$a_{n,1} = a_{n-1,1} + a_{n-2,1} + \cdots a_{1,1} + \underbrace{a_{0,1}}_{=1} + \underbrace{a_{0,n+1}}_{=1},$$

and thus we recursively find that $a_{1,1} = 2$, $q_{2,1} = 2 + 1 + 1 = 4$, $\cdots$, $a_{n,1} = 2^n$. For $n \geq Q$, we obtain:

$$\begin{aligned} a_{n,1} &= a_{n-1,1} + a_{n-1,2} \\ &= a_{n-1,1} + a_{n-2,1} + a_{n-2,3} \\ &= \cdots \\ &= a_{n-1,1} + a_{n-2,1} + \cdots + a_{n-Q+1,1}. \end{aligned}$$

Thus, the sequence is given by

$$a_{0,1} = 1, \quad a_{1,1} = 2, \quad \cdots, \quad a_{Q-1,1} = 2^{Q-1}, \quad \text{and} \quad a_{n,1} = \sum_{j=1}^{Q-1} a_{n-j,1} \quad \text{for } n \geq Q.$$

This is none other than the generalized Fibonacci sequence, where each term is the sum of the previous $Q-1$ terms!

Therefore, the maximal guaranteed rate of return in a game with N questions is therefore

$$r_{N,0} = r_{N-1,1} = \frac{2^n}{a_{N-1,1}} - 1.$$

FOr the base game with $N = 4$ and $Q = 3$, we have $a_{0,1} = 1$, $a_{1,1} = 2$, $a_{3,1} = a_{2,1} + a_{1,1} = 3$, and $a_{3,1} = a_{2,1} + a_{1,1} = 3 + 2 = 5$. Thue the maximal guaranteed rate of return is

$$r_{4,0} = \frac{2^3}{5} - 1 = \frac{3}{5}.$$

If we start with one dollar, we can always win an extra $\frac{3}{5}$ of a dollar and end up with $\frac{8}{5}$ dollars.